

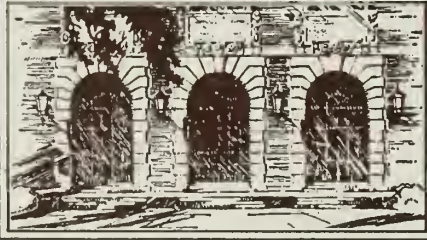
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GENERALIZED DECOMPOSITION OF INCOMPLETE FINITE AUTOMATA

by

Arthur Ta-shiang Pu

December 30, 1965

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GENERALIZED DECOMPOSITION OF INCOMPLETE FINITE AUTOMATA

by

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December 30, 1965

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This work was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, December, 1965.

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GENERALIZED DECOMPOSITION OF

INCOMPLETE FINITE AUTOMATA

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University of Illinois, 1966

The problem of decomposing a finite automaton has been investigated by many authors. However, their results were based on the question of decomposing an automaton into series and parallel connections of automata. The present work is an extension to the problem of generalized decomposition where two-way interconnections between automata are permitted. We are mainly concerned with decompositions for deterministic incomplete finite automata. Our decomposition does not presuppose the logical design of the circuit of an automaton.

With the new technology, the problem of economical realization no longer lies in the actual complexity of the logical design in each building block. Aside from a given upper limit, the complexity is not reflected in the cost. Subject to the restraint of the given limit on each block, our main object is to minimize the number of interconnections between blocks of a generalized decomposition of an automaton.

We first consider decomposing a given automaton $M = \langle S, T, f \rangle$ into a pair of automata $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$. We give the definition of a generalized pair decomposition (GPD) $[M_1 : M_2]$ of M . We define a $*$ -cover

$\mathcal{A} = (A_1, \dots, A_\ell)$ of S as an ℓ -tuple such that for each $i \in \{1, \dots, \ell\}$, $A_i \subseteq S$ and $\bigcup_{i=1}^{\ell} A_i = S$. We show that each GPD

of M gives rise to two $*$ -covers $\mathcal{A}, \mathcal{B}$ of S . We then establish

a necessary and sufficient condition for a pair of $*$ -covers (α, β) of S to be obtainable in this manner from a GPD of M .

In discussing the properties of a GPD $\{M_1:M_2\}$ of M , we show that all properties are represented by $\{M_1:M_2\}$ as a partition pair decomposition of an automaton $M'=\langle S', T, f' \rangle$, where M is a homomorphic image of M' . We introduce the notion of input configurations between component automata of a GPD and define the notion of connecting lines in terms of these input configurations. We deduce that to find the number of input configurations between component automata is actually a covering problem. In each GPD $\{M_1:M_2\}$ of M , we obtain a lower bound k for the number of input configurations from one component to the other. This lower bound k , obtained from certain properties of an equivalence relation, is much easier to compute than the solution of the covering problem. We then show that for certain GPDs of M , the number of input configurations from one component to the other actually achieves its lower bound k .

We then prove that the known results for series and parallel decomposition of an automaton are included in our results as special cases. We also raise the question of whether it is sufficient to consider just set systems in the general theory of decomposition for automata.

Finally, we extend our theory of generalized pair decomposition to generalized N -tuple decomposition (GND). We show that any GND of M can be obtained from successive pairwise decomposition.

INTRODUCTION

The theory of decomposition for finite automata has been investigated by many authors. Hartmanis [7,8,9], Stearns and Hartmanis [10] have published a series of articles using partitions with substitution property. Probing through the properties of semigroups, Krohn and Rhodes [15] obtain some very interesting results. Then Kohavi [14] and Hartmanis and Stearns [12] introduce the notion of set systems to take advantage of the state splitting. However, their results and the results of others [4,22,23] were based on the question of how to decompose an automaton into series or parallel connection of smaller automata. The information is not allowed to interflow between the component automata.

The question then arises as how does one consider decompositions for automata when information is allowed to interflow between the component automata?¹ This is a very important question. With the new technology, the problem of economical realization no longer lies in the complexity of logical design in each building block. Rather, they tend to depend on the number of connections between basic building blocks. The problem arising from future practical design thus will clearly emphasize the importance of having smaller numbers of connections.

With this in mind, the present work begins as an attempt to study the generalized decompositions of an automaton

¹We consider decomposing an automaton before it is actually constructed. This has the fundamental advantage that one may make adjustments accordingly in the actual design.

allowing interconnections. We will be mainly concerned with decompositions for a deterministic incomplete finite automaton. Our problem is to find, for a given incomplete automaton M , a generalized N -tuple decomposition $\{M_i\}_{1 \leq i \leq N}$ such that for each i , M_i has a complexity equal to or less than a given bound, and furthermore, the total number of interconnecting lines is a minimum over all such decompositions. Our results will include the known results for series and parallel decompositions as special cases.

In Chapter I, we first give the definition of generalized pair decomposition (GPD) $\{M_1:M_2\}$ of a given automaton $M = \langle S, T, f \rangle$, and the definition of generalized pair decomposition triplet (GPDT) $\langle S_1, S_2, \varphi \rangle$ of M . We show that each GPD $\{M_1:M_2\}$ of M naturally induces an automaton $G(M_1, M_2)$ realizing M . We then introduce the notion of $*$ -cover of a set S . A $*$ -cover of a set S is an ℓ -tuple $\alpha = (A_1, \dots, A_\ell)$, $\ell = \ell(\alpha)$ such that $(\forall i)(1 \leq i \leq \ell) [A_i \subseteq S]$ and $\bigcup_{i=1}^{\ell} A_i = S$. We show that each GPDT $\langle S_1, S_2, \varphi \rangle$ of M gives rise to a pair of $*$ -covers (α, β) . We then proceed to obtain a necessary and sufficient condition for a pair of $*$ -covers (α, β) of S to insure the existence of a function φ such that $\langle E_\alpha, E_\beta, \varphi \rangle$ is a GPDT of S . E_α, E_β are the index sets of α, β respectively. This work uses the added deleted distributive property.

In Chapter II, we introduce the definition of an incomplete monadic algebra $\Omega = \langle S, T, 0 \rangle$ and the definition of a congruence relation of degree k on Ω . We show that any incomplete, deterministic automaton can be considered as an incomplete monadic algebra, and conversely. We then find that the structure of a GPD $\{M_1:M_2\}$ of $M = \langle S, T, f \rangle$, where

$M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$, must be studied through the structure of $\{M_1:M_2\}$ as GPD of an automaton $M' = \langle S', T, f' \rangle$ where M is a homomorphic image of M' , and S_1, S_2 induces congruence relations of degree k_1, k_2 respectively. We introduce the notion of input configurations between M_1 and M_2 and show that k_1, k_2 are lower bounds on the number of input configurations from M_2 to M_1 and from M_1 to M_2 , respectively. We show that to find the number of input configurations is a covering problem.

Suppose $\mathcal{O}$ is a $*$ -cover of S and there exists an automaton $M' = \langle S', T, f' \rangle$ such that M' realizes M under φ . Suppose there also exists a non-trivial congruence relation θ_{11} of degree k_1 on M' such that θ_{11} induces $\mathcal{O}$ on S under φ . We show that for any θ_{22} on M' such that $\theta_{11} \cap \theta_{22} = 0$, θ_{11}, θ_{22} gives rise to a GPD $\{M_1:M_2\}$ of M . Finally, we show that there exists a non-trivial equivalence relation θ_{22} on S' such that if $\{M_1:M_2\}$ is the induced GPD of M under θ_{11} and θ_{22} , then the number of input configurations from M_2 to M_1 is exactly k_1 .

In section 1 of Chapter III, we show that the known results for series and parallel decompositions of an automaton are included in our results as special cases. A set system $\mathcal{O} = (A_1, \dots, A_p)$ of S is a special $*$ -cover of S in which $(\forall i, j) [(A_i \subseteq A_j) \iff (i=j)]$. In section 2, we exhibit examples to show that in a series or parallel decomposition it is not always possible to replace the pair of $*$ -covers by its induced set systems such that the resultant is still a series or parallel decomposition. We then raise the question whether it is sufficient just to consider set systems for decompositions of an automaton.

Finally, Chapter IV is the extension of the theory of GPD to the multiple case. We make modifications on the definitions and show that the results of pair decomposition do carry over to the multiple decomposition. We then show that any generalized multiple decomposition of an automaton can be obtained by successive usage of GPD. Thus, any discussion of multiple decomposition actually can be made through the discussion of pair decomposition.

Chapter I

PAIR DECOMPOSITION

We first introduce the following notations. Let E be a set, then $\mathcal{P}(E) = \{E_\alpha \mid E_\alpha \subseteq E\}$. Let $\{A_i \mid 1 \leq i \leq n\}$ be a collection of arbitrary finite sets of symbols. Let $\prod_{i=1}^n A_i$ denote the cartesian product of A_i . For each i , $1 \leq i \leq n$, define $\pi_i(a_1, \dots, a_n) = a_i \in A_i$. For each $\mathcal{S} \subseteq \prod_{i=1}^n A_i$, $\pi_i(\mathcal{S}) = \{\pi_i(a_1, \dots, a_n) \mid (a_1, \dots, a_n) \in \mathcal{S}\}$. If T is a finite set, then denote by T^* the free semigroup of all finite strings formed under concatenation. The empty string e is the identity element in T^* .

§0. Preliminaries

Definition 0.1. An automaton M (non-deterministic, incomplete, finite) is a triplet $\langle S, T, f \rangle$, where S is a finite set of "states", T is a finite set of "input letters", and f is a "transition function" mapping $S \times T$ into $\mathcal{P}(S)$.

Let $D(f) = \{(s, t) \mid f(s, t) \neq \emptyset\}$. Let $s \in S$ and $s \notin \{\pi_1(D(f)) \cup f(D(f))\}$. Then (i) $(\nexists t \in T)(f(s, t) \neq \emptyset)$, (ii) $(\nexists (s', t) \in S \times T)(f(s', t) = s)$. Therefore, we may choose the set S to consist of just the states in $\{\pi_1(D(f)) \cup f(D(f))\}$, and the set T to consist of just the input letters in $\pi_2(D(f))$. In fact, $D(f)$ is the normal notion for the domain of the function f .

The transition function f of an automaton can be

extended to the domain $S \times T^*$ by the following recursive rules. If we denote this extended function again by f , then $f(s, e) = s$ and $(\forall t \in T^*)(\forall t_1 \in T) \{f(s, tt_1) = \bigcup_{s' \in f(s, t)} f(s', t_1)\}$.

Definition 0.2. A deterministic automaton $M = \langle S, T, f \rangle$ is an automaton such that $\forall (s, t) \in D(f)$, $f(s, t)$ is a singleton.

Definition 0.3. A complete automaton $M = \langle S, T, f \rangle$ is an automaton such that $D(f) = S \times T$.

Definition 0.4. An automaton $M_2 = \langle S_2, T_2, f_2 \rangle$ is a subautomaton of $M_1 = \langle S_1, T_1, f_1 \rangle$ if and only if $T_2 \subseteq T_1$, $S_2 \subseteq S_1$, and $(\forall (s, t) \in S_2 \times T_2) [f_2(s, t) \subseteq f_1(s, t)]$.

Throughout the context of this dissertation by an automaton we shall always mean a deterministic incomplete finite automaton, unless otherwise stated.

Definition 0.5. An automaton $M_2 = \langle S_2, T_2, f_2 \rangle$ is a homomorphic image of an automaton $M_1 = \langle S_1, T_1, f_1 \rangle$ if and only if there exist an onto function $\varphi: S_1 \rightarrow S_2$, and an onto function $h: T_1 \rightarrow T_2$ such that $(\forall (s_1, t_1) \in (S_1 \times T_1)) [\varphi(f_1(s_1, t_1)) = f_2(\varphi(s_1), h(t_1))]$.

Definition 0.6. An automaton M_1 realizes another automaton M_2 if and only if M_2 is a homomorphic image of a subautomaton of M_1 .

In the case when φ and h are both one to one onto mappings, M_1 and M_2 are isomorphic.

§1. Generalized Pair Decomposition

Definition 1.1. Let $M = \langle S, T, f \rangle$ be a deterministic, incomplete automaton. Let $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ be two incomplete automata. Then $\{M_1, M_2\}$ is a generalized pair

decomposition (GPD) of M if and only if

(i) There exists a function φ defined on a subset

$$\mathcal{S} \subseteq S_1 \times S_2 \text{ onto } S \text{ such that } \pi_1(\mathcal{S}) = S_1, \\ \pi_2(\mathcal{S}) = S_2.$$

(ii) There exist two one to one functions h_1, h_2 , where

$$h_1: T_1 \longrightarrow T \times S_2, \quad h_2: T_2 \longrightarrow T \times S_1.$$

(iii) $D(f_1) \supseteq \{(s_1, t_1) \mid (\exists t \in T)[h_1(t_1) = (t, s_2)]$

$$[(\varphi(s_1, s_2), t) \in D(f)]\}$$

$$D(f_2) \supseteq \{(s_2, t_2) \mid (\exists t \in T)[h_2(t_2) = (t, s)]$$

$$[(\varphi(s_1, s_2), t) \in D(f)]\}$$

(iv) $(\forall (s_1, s_2) \in \mathcal{S})(\forall t)[(\varphi(s_1, s_2), t) \in D(f)] \implies$

$$\{(f_1(s_1, h_1^{-1}(t, s_2)), f_2(s_2, h_2^{-1}(t, s_1))) \in$$

$$\varphi^{-1}[f(\varphi(s_1, s_2), t)]\}$$

(v) Suppose $(s_1, s_2) \in D(\varphi)$ and $\varphi(s_1, s_2) = f(s, t)$ for some $s \in S, t \in T$. Then

$$[\exists (s_1^*, s_2^*) \in \varphi^{-1}(s)] \{ (s_1, s_2) = (f_1(s_1^*, h_1^{-1}(t, s_2^*)), \\ f_2(s_2^*, h_2^{-1}(t, s_1^*))) \}.$$

Let $\{M_1: M_2\}$ be a GPD of M . Define $\mathcal{D} \subseteq S_1 \times S_2$ such that $(s_1, s_2) \in \mathcal{D}$ if and only if either

(i) $\varphi(s_1, s_2) \in f(D(f))$, or

(ii) $(\exists t \in T) \{ [s_1, h_1^{-1}(t, s_2)] \in D(f_1) \} \{ [s_2, h_2^{-1}(t, s_1)] \in D(f_2) \}.$

Define a function F on a subset $D(F)$ of $\mathcal{D} \times T$ to $\mathcal{D}$ such that

$$[(s_1, s_2), t] \in D(F) \text{ if and only if } [(s_1, h_1^{-1}(t, s_2)) \in D(f_1)] \\ [(s_2, h_2^{-1}(t, s_1)) \in D(f_2)] \{ [f_1(s_1, h_1^{-1}(t, s_2)), f_2(s_2, h_2^{-1}(t, s_1))] \in \mathcal{D} \}.$$

For each $[(s_1, s_2), t] \in D(F)$, define $F[(s_1, s_2), t]$

$$= [f_1(s_1, h_1^{-1}(t, s_2)), f_2(s_2, h_2^{-1}(t, s_1))]. \text{ Certainly}$$

$\pi_1(D(F)) \cup F(D(F)) \subseteq \mathcal{D}$. Conversely, let $(s_1, s_2) \in \mathcal{D}$. If

$\varphi(s_1, s_2) \in f(D(f))$, then there exists $(s', t) \in D(f)$ such that $f(s', t) = \varphi(s_1, s_2)$. Then there exists $(s'_1, s'_2) \in \varphi^{-1}(s')$ such that $((s'_1, s'_2), t) \in D(F)$ and $F((s'_1, s'_2), t) = (s_1, s_2)$. Hence $(s_1, s_2) \in F(D(F))$. If for the pair (s_1, s_2) there exists $t \in T$ such that $[(s_1, h_1^{-1}(t, s_1)) \in D(f_1)] [(s_2, h_2^{-1}(t, s_2)) \in D(f_2)]$, then clearly $(s_1, s_2) \in \pi_1(D(F))$. Therefore, $\pi_1(D(F)) \cup F(D(F)) = \mathcal{D}$. It also can be easily shown that $\pi_2(D(F)) = T$. Thus, we arrive at the following proposition.

Proposition 1.1. The triplet $\langle \mathcal{D}, T, F \rangle$ is an incomplete automaton.

As a result, each GPD $[M_1:M_2]$ of M naturally induces a new incomplete automaton $\langle \mathcal{D}, T, F \rangle = G(M_1, M_2)$ called the generalized composition of $[M_1:M_2]$.

Since $\pi_1(D(f)) \cup f(D(f)) = S$, for each $(s_1, s_2) \in \mathcal{S} = D(\varphi)$ and then either $\varphi(s_1, s_2) \in f(D(f))$ or $(\exists t \in T) [(\varphi(s_1, s_2), t) \in D(f)]$, i.e., $(\exists t \in T) [(s_1, h_1^{-1}(t, s_2)) \in D(f_1)] [(s_2, h_2^{-1}(t, s_1)) \in D(f_2)]$. In either case, $(s_1, s_2) \in \mathcal{D}$. Hence $\mathcal{S} \subseteq \mathcal{D}$.

Proposition 1.2. Let $[M_1:M_2]$ be a GPD of M . Then the induced generalized composition of $[M_1:M_2]$; $G(M_1, M_2) = \langle \mathcal{D}, T, F \rangle$ realizes M .

Proof. We shall show that M is a homomorphic image of a subautomaton of $G(M_1, M_2)$. Since $[M_1:M_2]$ is a GPD of M , there exists a function φ from $\mathcal{S} \subseteq S_1 \times S_2$ onto S such that $\pi_1(\mathcal{S}) = S_1$, $\pi_2(\mathcal{S}) = S_2$. Let us consider the subautomaton $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ of $G(M_1, M_2) = \langle \mathcal{D}, T, F \rangle$. There exist an onto function $\varphi: \mathcal{S} \rightarrow S$, and an identity function $h: T \rightarrow T$ such that $(\forall (s_1, s_2, t) \in \mathcal{S} \times T) \{ \varphi(F(s_1, s_2, t)) = \varphi(f_1(s_1, h_1^{-1}(t, s_2)), f_2(s_2, h_2^{-1}(t, s_1))) \}$.

Since $f_1(s_1, h_1^{-1}(t, s_2)) \in \pi_1[\varphi^{-1}\{f(\varphi(s_1, s_2), t)\}]$ and $f_2(s_2, h_2^{-1}(t, s_1)) \in \pi_2[\varphi^{-1}\{f(\varphi(s_1, s_2), t)\}]$, thus $\varphi[f_1(s_1, h_1^{-1}(t, s_2), f_2(s_2, h_2^{-1}(t, s_1)))] = f(\varphi(s_1, s_2), t)$. Therefore $(\forall (s_1, s_2, t) \in \mathcal{S} \times T) [\varphi(F((s_1, s_2), t)) = f(\varphi(s_1, s_2), t)]$. Hence M is a homomorphic image of the submachine $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ of $G(M_1, M_2)$, and $G(M_1, M_2)$ realizes M .

Q. E. D.

We note that if $\{M_1, M_2\}$ is a GPD where $M_1 = \langle S_1, T_1, f_1 \rangle$ and $M_2 = \langle S_2, T_2, f_2 \rangle$, then there is an associated triplet $\langle S_1, S_2, \varphi \rangle$. Conversely, we have the following theorem.

Theorem 1.1. Let $M = \langle S, T, f \rangle$. Let $\langle S_1, S_2, \varphi \rangle$ be a triplet where φ is a function from $\mathcal{S} \subseteq S_1 \times S_2$ onto S such that $\pi_1(\mathcal{S}) = S_1$, $\pi_2(\mathcal{S}) = S_2$. Then there exists a non-empty collection of GPDs of M whose associated triplet is the given $\langle S_1, S_2, \varphi \rangle$.

Proof. Define two automata, $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ as follows:

- (i) $T_1 = T \times S_2$ $T_2 = T \times S_1$
- (ii) $(\forall s_1 \in S_1)(\forall s_2 \in S_2)(\forall t \in T) \{ [f_1(s_1, (t, s_2)), f_2(s_2, (t, s_1))] \in \varphi^{-1}\{f(\varphi(s_1, s_2), t)\} \}$.

Since $f(\varphi(s_1, s_2), t)$ makes sense and is non-empty if and only if $(s_1, s_2) \in \mathcal{S}$, certainly

$$\begin{aligned} D(f_1) &= \{(s_1, (t, s_2)) \mid (s_1, s_2) \in \mathcal{S}\} \\ D(f_2) &= \{(s_2, (t, s_1)) \mid (s_1, s_2) \in \mathcal{S}\}. \end{aligned}$$

Therefore, $\{M_1, M_2\}$ is a GPD of M and its associated triplet is $\langle S_1, S_2, \varphi \rangle$. Furthermore, the set $\{\varphi^{-1}\{f(\varphi(s_1, s_2), t)\}\}$ may contain more than one element. For each choice of $f_1(s_1, (t, s_2))$ and $f_2(s_2, (t, s_1))$ in $\varphi^{-1}\{f(\varphi(s_1, s_2), t)\}$, there exists a GPD of M .

Thus there may be more than one GPD defined this way. However, clearly each such GPD has its associated triplet being

$$\langle S_1, S_2, \varphi \rangle.$$

Q. E. D.

We call such a triplet $\langle S_1, S_2, \varphi \rangle$, a generalized pair decomposition triplet (GPDT) of M.

Definition 1.2. Let $M = \langle S, T, f \rangle$. Let $M_1 = \langle S_1, T_1, f_1 \rangle$ and

$M_2 = \langle S_2, T_2, f_2 \rangle$ such that $\{M_1 : M_2\}$ is a GPD of M. Define a pair of non-deterministic machines $\bar{M}_1 = \langle S_1, T, F_1 \rangle$ and $\bar{M}_2 = \langle S_2, T, F_2 \rangle$ as follows:

- (i) $(\forall s_1 \in S_1)(\forall t \in T) \{ F_1(s_1, t) = \{ s_1' \in S_1 \mid (\exists s_2 \in S_2) s_1' \in \pi_1[\varphi^{-1}[f(\varphi(s_1, s_2), t)]] \} \}$
- (ii) $(\forall s_2 \in S_2)(\forall t \in T) \{ F_2(s_2, t) = \{ s_2' \in S_2 \mid (\exists s_1 \in S_1) s_2' \in \pi_2[\varphi^{-1}[f(\varphi(s_1, s_2), t)]] \} \}.$

The pair $(\bar{M}_1, \bar{M}_2)$ so defined is called the pair of components of this GPD $\{M_1 : M_2\}$.

Proposition 1.3. Let $\langle S_1, S_2, \varphi \rangle$ be a GPDT of M. Let $\{M_1 : M_2\}$ and $\{M_3 : M_4\}$ be two GPD of M with the same GPDT $\langle S_1, S_2, \varphi \rangle$. Then the components of $\{M_1 : M_2\}$ are the same as those of $\{M_3 : M_4\}$.

Proof. For each $i, 1 \leq i \leq 4$, let $\bar{M}_i = \langle S_i, T_i, F_i \rangle$ be the components of $\{M_1 : M_2\}$ and $\{M_3 : M_4\}$. Since these two GPD have the same associated triplet, $S_1 = S_3, S_2 = S_4$ and $T = T_1 = T_2 = T_3 = T_4$. It remains to show that $F_1 = F_3$ and $F_2 = F_4$. This is quite obvious, since the transition function depends solely on the triplet $\langle S_1, S_2, \varphi \rangle$.

Q. E. D.

As the result of the above proposition, there exists a unique pair of non-deterministic automata associated with each GPDT of M.

Definition 1.3. The unique pair of non-deterministic automata associated with each GPDT is called the pair of components of the GPDT.

Proposition 1.4. Let $\{M_1:M_2\}$ be a GPD of M . If $(\bar{M}_1, \bar{M}_2)$ is the pair of components of $\{M_1:M_2\}$, then for each i , $1 \leq i \leq 2$, M_i is a subautomaton of $\bar{M}_i$.

Proof. Consider deterministic automata being a special case of non-deterministic automata in the usual way. The proof is obvious from the definition.

Q.E.D.

§2. *-covers and Generalized Pair Decomposition Triplet

For any given set A , a cover $\mathcal{C}$ of A is a collection of non-empty subsets A_k of A such that their union is A . We introduce the definition of a *-cover of a given set.

Definition 2.1. Let $\bar{S}$ be a finite set of arbitrary symbols.

Then $\mathcal{A}$ is called a *-cover of $\bar{S}$ if there exists a positive integer $\ell = \ell(\mathcal{A})$ such that (i) $\mathcal{A}$ is an ℓ -tuple

$(A_1, \dots, A_\ell)$ where each A_i is a non-empty subset of $\bar{S}$,

(ii) $\bigcup_{i=1}^{\ell} A_i = \bar{S}$.

Definition 2.2. Let $\bar{\mathcal{A}} = (A_1, \dots, A_\ell)$ be a *-cover of a finite set of symbols $\bar{S}$. Let ρ be a many to one function from $\bar{S}$ onto S where S is another set. Let $\mathcal{A} = (\rho(A_1), \dots, \rho(A_\ell))$ where $(\forall i)(1 \leq i \leq \ell)[\rho(A_i) = \{\rho(\sigma) \mid \sigma \in A_i\}]$. Then $\mathcal{A}$ is a *-cover of S and is said to be induced by the *-cover $\bar{\mathcal{A}}$ of $\bar{S}$ under ρ .

Two *-covers $\mathcal{A}, \mathcal{B}$ of S are said to be equivalent if $\mathcal{A}$ can be obtained from $\mathcal{B}$ by permuting the coordinates of $\mathcal{B}$.

Our next theorem immediately reveals the importance of $*$ -covers.

Theorem 2.1. For each GPDT $\langle S_1, S_2, \varphi \rangle$ of $M = \langle S, T, f \rangle$, and each ordering of states in $S_1 = \{a_1^1, a_2^1, \dots, a_\ell^1\}$, and $S_2 = \{a_1^2, \dots, a_m^2\}$ we have that there exists a pair of $*$ -covers $\mathcal{A}_1 = (A_1^1, \dots, A_\ell^1)$ and $\mathcal{A}_2 = (A_1^2, \dots, A_m^2)$ of S . Furthermore, for each i , $1 \leq i \leq \ell$, $A_i^1 = \{s \in S \mid (\exists k)(\varphi(a_i^1, a_k^2) = s)\}$, and for each j , $1 \leq j \leq m$, $A_j^2 = \{s \in S \mid (\exists k)(\varphi(a_k^1, a_j^2) = s)\}$.

Proof. Since $\langle S_1, S_2, \varphi \rangle$ is a GPDT, we can represent the function φ by a $\ell \times m$ table as follows: The rows are labelled by a_i^1 , $1 \leq i \leq \ell$, the columns are labelled by a_j^2 , $1 \leq j \leq m$. In the entry (i, j) , we put the state s , if $(a_i^1, a_j^2) \in D(\varphi)$ and $\varphi(a_i^1, a_j^2) = s$. For each i , $1 \leq i \leq \ell$, let A_i^1 be the set containing the elements in the i^{th} row. For each j , $1 \leq j \leq m$, let A_j^2 be the set containing the elements in the j^{th} column. Certainly A_i^1, A_j^2 have the desired property. Let $\mathcal{A}_1 = (A_1^1, \dots, A_\ell^1)$, $\mathcal{A}_2 = (A_1^2, \dots, A_m^2)$. To show $\mathcal{A}_1, \mathcal{A}_2$ are $*$ -covers of S , we need only show that $\bigcup_{i=1}^{\ell} A_i^1 = S$ and $\bigcup_{j=1}^m A_j^2 = S$. But this is readily so, since φ is an onto function.

Q. E. D.

We note that any reordering of S_1 and S_2 will give rise to $*$ -covers $\mathcal{A}_1', \mathcal{A}_2'$ which are equivalent to $\mathcal{A}_1, \mathcal{A}_2$ respectively. Hence we can consider S_1, S_2 as two ordered sets.

We shall now try to establish a theorem (Theorem 2.4) in the other direction, namely a necessary and sufficient condition for a pair of $*$ -covers $(\mathcal{A}, \mathcal{B})$ of S to insure the existence of a function φ such that $\langle S_1, S_2, \varphi \rangle$ is a GPDT of M , where S_1, S_2 give rise to $\mathcal{A}, \mathcal{B}$ respectively. To do this, we first obtain a weaker statement (Theorem 2.2). Using that as a step-

ping stone, we arrive at our desired result.

Definition 2.4. Let $\mathcal{A} = (A_1, \dots, A_\ell)$, $\mathcal{B} = (B_1, \dots, B_m)$ be $*$ -covers of the finite set S . Then $(\mathcal{A}, \mathcal{B})$ is said to be mutually distributive (MD) if and only if

- (i) For each coordinate A_i of $\mathcal{A}$, where $A_i = \{s_1, \dots, s_p\}$, there exists at least one subset $\{i_1, \dots, i_p\}$ of $\{1, 2, \dots, m\}$ such that $s_j \in B_{i_j}$ for each j , $1 \leq j \leq p$.
- (ii) A symmetric condition is satisfied for each coordinate B_i of $\mathcal{B}$.

To serve as an example, let $S = \{s_1, s_2, \dots, s_5\}$.

Let $\mathcal{A} = (A_1 = \{s_1, s_2, s_3\}, A_2 = \{s_3, s_4\}, A_3 = \{s_2, s_4, s_5\})$,
 $\mathcal{B} = (B_1 = \{s_1, s_5\}, B_2 = \{s_2, s_3, s_4\}, B_3 = \{s_2, s_3, s_4\})$. Then the pair $(\mathcal{A}, \mathcal{B})$ is MD.

Definition 2.5. Let $\mathcal{A}$ be a $*$ -cover of S . For each $s \in S$, the multiplicity of s in $\mathcal{A}$, denoted by $m_{\mathcal{A}}(s)$, is the number of coordinates in $\mathcal{A}$ containing s .

Definition 2.6. Let $\mathcal{A} = (A_1, \dots, A_\ell)$ be a $*$ -cover of S . Let $s \in S$, then for any i , $1 \leq i \leq \ell \ni s \in A_i$, $\mathcal{A}_{(s,i)}$ is defined as the ℓ -tuple $(A_1, \dots, A_{i-1}, A_i - (s), A_{i+1}, \dots, A_\ell)$.

In the following definitions, $\mathcal{A} = (A_1, \dots, A_\ell)$,
 $\mathcal{B} = (B_1, \dots, B_m)$.

Definition 2.7. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of S such that $(\forall s \in S)[m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s) = m(s)]$. For each $s \in S \ni m(s) > 1$, $(\mathcal{A}, \mathcal{B})$ is said to have the 1st s -deleted distributive property if

- (i) For each i , $1 \leq i \leq \ell$, $\exists s \in A_i$, $\exists j$, $1 \leq j \leq m \ni s \in B_j$ and $(\mathcal{A}_{(s,i)}, \mathcal{B}_{(s,j)})$ is MD.
- (ii) For each j , $1 \leq j \leq m \ni s \in B_j$, $\exists i$, $1 \leq i \leq \ell \ni s \in A_i$ and $(\mathcal{A}_{(s,i)}, \mathcal{B}_{(s,j)})$ is MD.

Definition 2.8. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of S such that $(\forall s \in S) [m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s) = m(s)]$. For each $s \in S$, $m(s) > 1$, $(\mathcal{A}, \mathcal{B})$ is said to have the n^{th} -s-deleted distributive property if

(i) For each i , $1 \leq i \leq n \ni s \in A_i$, $\exists j$, $1 \leq j \leq m \ni s \in B_j$ and $(\mathcal{A}_{(s,i)}, \mathcal{B}_{(s,j)})$ has the $(n-1)^{\text{th}}$ -s-deleted distributive property.

(ii) For each j , $1 \leq j \leq m \ni s \in B_j$, $\exists i$, $1 \leq i \leq n \ni s \in A_i$ and $(\mathcal{A}_{(s,i)}, \mathcal{B}_{(s,j)})$ has the $(n-1)^{\text{th}}$ -s-deleted distributive property.

Definition 2.9. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of S such that $(\forall s) [m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s) = m(s)]$. Then $(\mathcal{A}, \mathcal{B})$ is said to have the deleted distributive property (DDP) if for each $s \in S \ni m(s) > 1$, $(\mathcal{A}, \mathcal{B})$ has the $(m(s)-1)^{\text{th}}$ -s-deleted distributive property.

The pair of $*$ -covers in the example following Definition 2.4 has the DDP. The following example will show that not all MD pairs of $*$ -covers of S in which $(\forall s) [m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s) = m(s)]$ will have the DDP. Let $S = \{s_1, \dots, s_5\}$. Let $\mathcal{A} = (A_1 = \{s_1, s_2, s_3\}, A_2 = \{s_2, s_3, s_4\}, A_3 = \{s_4, s_5\})$, $\mathcal{B} = (B_1 = \{s_1, s_2, s_4\}, B_2 = \{s_2, s_3, s_4\}, B_3 = \{s_3, s_5\})$. It is easily checked that they are MD and $(\forall s) [m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s)]$. Yet $(\mathcal{A}, \mathcal{B})$ does not have DDP.

Definition 2.10. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of S .

Let $A_i = \{s_1, \dots, s_p\}$ be a coordinate of $\mathcal{A}$. Let $s_k \in A_i$. Then A_i is said to require s_k in a coordinate B_{k_0} of $\mathcal{B}$ to distribute if whenever $\{i_1, \dots, i_p\} \subseteq \{1, 2, \dots, m\}$ satisfies the condition (i) in Definition 2.4, then $i_k = k_0$.

Definition 2.11. Let $\mathcal{A} = (A_1, \dots, A_\ell)$ be a $*$ -cover of S . By

E_{α} , we shall mean the set of indices of α , i.e.,

$$E_{\alpha} = \{1, 2, \dots, \ell\}.$$

In the following, let $\alpha = (A_1, \dots, A_{\ell})$,

$$\beta = (B_1, \dots, B_m).$$

Lemma 2.1. Let (α, β) be a MD pair of $*$ -covers of S

$\ni (\forall s \in S) [m_{\alpha}(s) = m_{\beta}(s) = m(s)]$. Let E_{α} , E_{β} be defined as above. Then there exists a function φ defined on a subset

$E_{\beta} \subseteq E_{\alpha} \times E_{\beta}$ onto S such that

- (i) $(\forall i)(1 \leq i \leq \ell)(A_i = \{s | (\exists j \in E_{\beta}) [\varphi(i, j) = s]\})$
- (ii) $(\forall j)(1 \leq j \leq m)(B_j = \{s | (\exists i \in E_{\alpha}) [\varphi(i, j) = s]\})$
- (iii) For each $s \in S$, define $D_s(\varphi) = \{(i, j) \in E_{\alpha} \times E_{\beta} | \varphi(i, j) = s\}$;
then the cardinality of $D_s(\varphi)$ is $m(s)$.

if and only if there exists a rectangular table with columns labeled by j , $1 \leq j \leq m$, and rows labeled by i , $1 \leq i \leq \ell$, such that

- (i) Each $s \in S$ appears in the table exactly $m(s)$ times and in exactly $m(s)$ columns and $m(s)$ rows.
- (ii) Each column j , $1 \leq j \leq m$, gives an exact list of the elements in the j^{th} coordinates B_j of β . Each i^{th} row, $1 \leq i \leq \ell$ gives an exact list of the elements in the i^{th} coordinates A_i of α .

Proof. The "if" part is trivial. Simply define $\varphi(i, j) = s$, if s occupies the entry (i, j) . Conversely, suppose there exists a φ satisfying (i), (ii), (iii). Assign to each (i, j) of $E_{\beta} \subseteq E_{\alpha} \times E_{\beta}$ the element $\varphi(i, j)$. Since φ is a function, the assignment is unambiguous for each entry. Further, it is clear that the table obtained satisfies the desired properties.

Q. E. D.

Lemma 2.2. Let (α, β) be a MD pair of $*$ -covers of S . Let

$s, s' \in S$, $s \neq s'$ and $m_{\alpha}(s) = m_{\beta}(s) = 1$, $m_{\alpha}(s') = m_{\beta}(s') = 1$.

Then there does not exist (i, j) where $i \in \{1, 2, \dots, \ell\}$,

$j \in \{1, \dots, m\}$ such that $\{s, s'\} \subseteq A_i \cap B_j$.

Proof. Suppose there exists (i, j) , $i \in \{1, \dots, \ell\}$, $j \in \{1, \dots, m\}$ such that $\{s, s'\} \subseteq A_i \cap B_j$. Since $m_{\alpha}(s) = m_{\mathcal{B}}(s') = m_{\mathcal{B}}(s) = m_{\alpha}(s') = 1$, B_j is the only coordinate of $\mathcal{B}$ containing s and s' . Further, since $A_i \supseteq \{s, s'\}$, A_i can not be distributive over $\mathcal{B}$. This contradicts the MD property of $(\alpha, \mathcal{B})$.

Q.E.D.

Lemma 2.3. Let $(\alpha, \mathcal{B})$ be a MD pair of $*$ -covers of S . Let A_i be a coordinate of α . Let $s \in A_i$ and A_i requires s in a coordinate B_k of $\mathcal{B}$ to distribute. Suppose $s' \in A_i \cap B_k$ and $s' \neq s$. Let $A_i = \{s_1, \dots, s_p\}$ where $s_1 = s$, $s_2 = s'$. For each subset $\{i_1, \dots, i_p\}$ of $\{1, 2, \dots, m\}$ such that $(\forall j)(1 \leq j \leq p)(s_j \in B_{i_j})$, we have that $i_1 = k$ and $s = s_1 \notin B_{i_2}$.

Proof. Suppose there exists a subset $\{i_1, \dots, i_p\}$ of $\{1, \dots, m\}$ such that $(\forall j)(1 \leq j \leq p)(s_j \in B_{i_j})$ and that $s \in B_{i_2}$. Then certainly the subset $\{i_1', \dots, i_p'\}$ of $\{1, 2, \dots, m\}$, where $i_1' = i_2$, $i_2' = i_1$, and $i_j' = i_j$ for $3 \leq j \leq p$, will have the same property. Furthermore, $i_1' \neq k$, since $i_2' = i_1 = k$ and $i_2' = i_1 \neq i_1' = i_2$. This contradicts the assumption that A_i requires s in B_k to distribute.

Q.E.D.

Lemma 2.4. Let $(\alpha, \mathcal{B})$ be a MD pair of $*$ -covers of S . Let A_i be a coordinate of α $\ni s \in A_i$ and A_i requires s in a coordinate B_j of $\mathcal{B}$ to distribute. Suppose $s' \in A_i \cap B_j$, $s' \neq s$, then A_i can not also require s' in B_j to distribute. Furthermore, if $(\alpha, \mathcal{B})$ has DDP, then B_j can not require s' in A_i to distribute.

Proof. Let $A_i = \{s_1, \dots, s_p\}$ and $s = s_h$, $s' = s_k$, $h \neq k$. Since A_i requires s in B_j to distribute, $\exists \{i_1, \dots, i_p\} \subseteq \{1, 2, \dots, m\}$ such that $(\forall j)(1 \leq j \leq p)(s_j \in B_{i_j})$. Suppose A_i also requires s'

in B_j to distribute. Then $i_h = i_k = j$ which is a contradiction.

Suppose $(\mathcal{A}, \mathcal{B})$ has DDP and B_j requires s' in A_i to distribute. Then B_j does not use s in A_i to distribute. Let $B_j = \{s_1, \dots, s_q\}$ and let $s = s_h$, $s' = s_k$, $k \neq h$. Let $\{j_1, \dots, j_q\}$ be the subset of $\{1, \dots, \ell\}$ such that $(\forall \alpha)(1 \leq \alpha \leq q)(s_\alpha \in A_{j_\alpha})$. Then $j_k = i$ and $j_h \neq i$. Since $(\mathcal{A}, \mathcal{B})$ has DDP, for the set A_{j_h} , there exists $j_0 \in \{1, \dots, m\} \ni (\mathcal{A}', \mathcal{B}') = (\mathcal{A}_{(s, j_h)}, \mathcal{B}_{(s, j_0)})$ also has DDP. Yet A_i requires s in B_j to distribute and $j_h \neq i$, thus $j_0 \neq j$. Therefore, B_j does not require s in any coordinate A_r , $r \neq i$, of $\mathcal{A}$ to distribute.

Now, consider the new pair $(\mathcal{A}', \mathcal{B}')$. The conditions of this lemma still apply. The multiplicity of s in $\mathcal{A}'$ or $\mathcal{B}'$, denoted by $m'(s)$ is $(m(s) - 1)$. As long as $m'(s) > 2$, B_j does not require s in any A_r , $r \neq i$, of $\mathcal{A}'$ to distribute for the same reason. We may similarly arrive at a new pair $(\mathcal{A}'', \mathcal{B}'')$. Suppose $m''(s) > 2$, we proceed to obtain $(\mathcal{A}''', \mathcal{B}''')$ and so forth until $m^{(\gamma)}(s) = 2$. Then, since $(A_i \cap B_j)$ still contains s , there is only one other A' of $\mathcal{A}^{(\gamma)} \ni s \in A'$. By a similar argument, B_j does not require s in any A_r , $r \neq i$ of $\mathcal{A}^{(\gamma)}$ to distribute. However, B_j can not use s in A_i to distribute and there is only one other A' containing s , B_j must require s in A' to distribute. This contradicts the fact that B_j does not require any A_r , $r \neq i$, to distribute. Therefore, B_j does not require s' in A_i to distribute.

Q. E. D.

Lemma 2.5. Let $(\mathcal{A}, \mathcal{B})$ be a pair of $*$ -covers of S with DDP. Let $i \in \{1, 2, \dots, \ell\}$ such that $s \in A_i$ and A_i requires s in a coordinate B_k of $\mathcal{B}$ to distribute. Then for any $j \neq i$, $j \in \{1, \dots, \ell\}$, A_j can not require s in B_k to distribute. Furthermore, for any $r \neq k$, $r \in \{1, \dots, m\}$, B_r can not require s in

A_i to distribute.

Proof. Suppose $\exists j \neq i, j \in \{1, \dots, \ell\}, A_j$ also requires s in B_k to distribute. Then for $s \in B_k$, there does not exist any coordinate A_α of $\mathcal{A}$ such that $s \in A_\alpha$, and $(\mathcal{A}_{(s, \alpha)}, \mathcal{B}_{(s, k)})$ again has DDP. This is a contradiction.

Suppose there exists $r \neq k$ such that B_r requires s in A_i to distribute. Then $(\mathcal{A}_{(s, i)}, \mathcal{B}_{(s, k)})$ can not have DDP. However, since A_i requires s in B_k to distribute, for s in B_k , B_k is the only coordinate of $\mathcal{B}$ from which s can be deleted together with s in A_i to give a new pair of $*$ -covers with DDP. Thus, if B_r requires s in A_i to distribute, then $(\mathcal{A}, \mathcal{B})$ can not have DDP. Since for s in B_k , there does not exist any coordinate A_α of $\mathcal{A}$ such that $s \in A_\alpha$ and $(\mathcal{A}_{(s, \alpha)}, \mathcal{B}_{(s, k)})$ has the $(m(s)-1)^{\text{th}}$ - s -deleted distributive property.

Q. E. D.

Remark 2.1. Let $s \in S$. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of $S \ni m_{\mathcal{A}}(s) = m_{\mathcal{B}}(s) = 1$. Let A, B be the only coordinate in $\mathcal{A}, \mathcal{B}$ respectively containing s . Then A requires s in B to distribute and B requires s in A to distribute.

Lemma 2.6. Let $(\mathcal{A}, \mathcal{B})$ be a pair of $*$ -covers of S with DDP. Let A_i be a coordinate of $\mathcal{A}$. Suppose $s \in A_i$ such that (i) A_i does not require s in any coordinate B_j of $\mathcal{B}$ to distribute, (ii) For any B_j of $\mathcal{B} \ni s \in B_j$, B_j does not require s in A_i to distribute. Then there exists $j_0, 1 \leq j_0 \leq m \ni s \in B_{j_0}$ (1) B_{j_0} does not require s in any coordinate A_r of $\mathcal{A}$ to distribute, (2) Suppose $s' \in S, s' \neq s$ and $s' \in A_i \cap B_{j_0}$, then A_i does not require s' in B_{j_0} to distribute and B_{j_0} does not require s'

in A_i to distribute.

Proof. Let $(\mathcal{A}, \mathcal{B})$ be given and that $s \in A_i$ of $\mathcal{A}$ satisfying the conditions (i), (ii). Suppose for every coordinate B_j of $\mathcal{B} \ni s \in B_j$, B_j requires s in some coordinate A_r of $\mathcal{A}$ to distribute. Then, by (ii), $r \neq i$. There are only $m(s) - 1$ coordinates A_r in which $s \in A_r$ and $r \neq i$. Yet there are $m(s)$ coordinates of $\mathcal{A}$ containing s . By Lemma 2.5, there should be $m(s)$ distinct coordinates A_r such that $s \in A_r$ and $r \neq i$. This leads to a contradiction. Therefore the set $J = \{j \mid j \in \{1, \dots, m\}, s \in B_j, B_j \text{ does not require } s \text{ in any coordinate } A_r \text{ of } \mathcal{A} \text{ to distribute}\} \neq \emptyset$. We note that for any $j \in J$, j satisfies the conclusion (1).

To show the existence of j_0 satisfying (1), (2), we need only show the existence of $j_0 \in J$ such that j_0 satisfies (2). Suppose there does not exist any $j_0 \in J = \{j_1, \dots, j_p\}$ satisfying (2). Then there exists $\{s_{j_1}, \dots, s_{j_p}\}$ where $s_{j_k} \neq s$ for any k , $1 \leq k \leq p$, with the property that either B_{j_k} requires s_{j_k} in A_i to distribute or A_i requires s_{j_k} in B_{j_k} to distribute. Suppose $s = s_1$ in A_i , then $j_k \neq 1$ for each k , thus, by Lemma 2.3, Lemma 2.4, for any $j_k \in J$, and any subset $\{i_1, \dots, i_q\}$ of $\{1, 2, \dots, \ell\}$ satisfying Definition 2.4, $i_1 \neq j_k$. We shall say that A_i must require $s = s_{i_1}$ to be not in B_{j_k} to distribute. But, by definition of J , for all $k \in J^* = \{k \mid (1 \leq k \leq m)(s \in B_k)(k \notin J)\}$, B_k requires s in some coordinate A_{r_k} to distribute, where $r_k \neq i$. Let $(\mathcal{A}', \mathcal{B}')$ be the $*$ -covers of S obtained from deleting s in B_k and A_{r_k} for each $k \in J^*$. Then $(\mathcal{A}', \mathcal{B}')$ must still have DDP, and we have $(s \in A_i)((s \in B_j) \iff (j \in J))$ where A_i, B_j are the i^{th} and j^{th} entry of $\mathcal{A}', \mathcal{B}'$ respectively.

Furthermore, for each $j \in J$, A_i still requires s to be not in any B_j to distribute. But by the construction of $\mathcal{B}'$, there exists no coordinate B_k of $\mathcal{B}'$, $k \notin J$ which contains s . Thus, A_i is not distributive over $\mathcal{B}'$, a contradiction to the fact that $(\mathcal{A}', \mathcal{B}')$ has DDP. Therefore, there must exist a $j_0 \in J$ $\ni B_{j_0}$ does not require s' in A_i to distribute and A_i does not require s' in B_j to distribute for any $s' \neq s$. This proves the lemma.

Definition 2.12. Let $\mathcal{A} = (A_1, \dots, A_r)$, $\mathcal{B} = (B_1, \dots, B_m)$. Let $(\mathcal{A}, \mathcal{B})$ be a MD pair of $*$ -covers of S . Let A_k be the k^{th} coordinate of $\mathcal{A} \ni s \in A_k$. Let B_h be the h^{th} coordinate of $\mathcal{B} \ni s \in B_h$. Let $A_k = \{s_1, s_2, \dots, s_i = s, s_{i+1}, \dots, s_p\}$. Let $\{n_1, \dots, n_r\} \subseteq \{1, \dots, p\}$ be such that $n_j \neq i$ for each j , $1 \leq j \leq r$. Let α be a one to one function from $\{n_j | 1 \leq j \leq r\}$ into $\{1, 2, \dots, m\}$. Then A_k is said to require $s_i = s$ in B_h to distribute with fixed $B_{\alpha(n_1)}, \dots, B_{\alpha(n_r)}$ if and only if whenever $\{k_1, \dots, k_p\}$ is a subset of $\{1, 2, \dots, m\} \ni (\forall j) (1 \leq j \leq p) (s_j \in B_{k_j})$, and for each q , $1 \leq q \leq r$, $k_{n_q} = \alpha(n_q)$, then $k_i = h$.

Remark 2.2. The statement " A_k (or B_k) requires s_i in B_h (or A_h) to distribute" and the statement " A_k (or B_k) requires s_i in B_h (or A_h) to distribute after fixing $B_{\alpha(n_1)}, \dots, B_{\alpha(n_r)}$ (or $A_{\alpha(n_1)}, \dots, A_{\alpha(n_r)}$)" are the same when $m(s_i) = 1$.

Remark 2.3. The results of Lemma 2.2. to Lemma 2.6 are still true when one replaces the statement " A_k (or B_h) requires s_i in B_h (or A_k) to distribute" by the statement " A_k (or B_h) requires s_i in B_h (or A_k) to distribute after fixing $B_{\alpha(n_1)}, \dots, B_{\alpha(n_r)}$ (or $A_{\alpha(n_1)}, \dots, A_{\alpha(n_r)}$)". Suppose $m(s_i) = 1$, then the result follows naturally. Thus we may assume $m(s_i) > 1$. Then first, whenever $m(s_{n_j}) > 1$, $1 \leq j \leq r$, simply delete s_{n_j} from

A_k and $\{s_{n_j}, s_i\}$ from $B_{\alpha(n_j)}$ and s_i in their corresponding $(A_{\alpha(n_j)})'$ such that after the deletion, the new $*$ -covers of S have DDP. Since A_k can be distributed without using the s_i in $B_{\alpha(n_1)}, \dots, B_{\alpha(n_r)}$, certainly $(A_{\alpha(n_j)})'$ can always be chosen not to denote A_k . Let (α', β') be the new pair of $*$ -covers after all such deletion, then $s_i \in A_k \cap B_h$. Now we can apply Lemma 2.2 to Lemma 2.6 to (α', β') to obtain the desired results.

Theorem 2.2. Let $\alpha = (A_1, \dots, A_\ell)$, $\beta = (B_1, \dots, B_m)$ be a MD pair of $*$ -covers of S . Let $E_\alpha = \{1, 2, \dots, \ell\}$, $E_\beta = \{1, 2, \dots, m\}$ be as defined in Definition 2.11. Then (α, β) has DDP if and only if there exists a function φ defined on a subset E_j of $E_\alpha \times E_\beta$ onto S such that

- (i) For each coordinate A_i of α , $A_i = \{s \in S \mid \exists j \in E_\beta \ni \varphi(i, j) = s\}$.
- (ii) For each coordinate B_j of β , $B_j = \{s \in S \mid \exists i \in E_\alpha \ni \varphi(i, j) = s\}$.
- (iii) For each $s \in S$, the number of elements in the set $\{(i, j) \mid \varphi(i, j) = s\}$ is $m(s)$.

Proof. For the only if part, by Lemma 2.1, we need only show that there exists a $(\ell \times m)$ rectangular table, not necessarily all filled, with each column labeled by i , giving the exact list of the elements in the coordinate A_i of α , and each row labeled by j giving the exact list of the elements in the coordinate B_j of β .

We shall describe a process in assigning states into the entries of the table. We will show that after each step the assigned state in the entry (i, j) is in $A_i \cap B_j$ and such that no one state repeatedly appears in the same row or same

column. We will then show that this process is exhaustive; i.e., every $s \in S$ is assigned to exactly $m(s)$ squares. And certainly we must show that each step described in the process is possible to complete.

- (I) (a) For each column j , assign $s \in B_j$ to the entry (i, j) if B_j requires s in A_i to distribute.
 (b) For each row i , assign $s \in A_i$ to the entry (i, j) if A_i requires s in B_j to distribute.

This process is possible to complete. As a result of Lemma 2.4, any coordinate A_i of $\mathcal{A}$ can not require two different $s, s' \in S$ in the same B_j to distribute. Again, by Lemma 2.4, if A_i requires s in B_j to distribute, then B_j can not require any other $s' \neq s$ in A_i to distribute. Thus (b) can be done without conflicting with (a), and vice versa.

Moreover, if A_i requires s in B_j to distribute and $(\mathcal{A}, \mathcal{B})$ is MD, by Lemma 2.5, we have that A_i does not require s in any other $B_k, k \neq j$, of $\mathcal{B}$ to distribute, and for each B_k of $\mathcal{B}, k \neq j$, B_k can not require s in A_i to distribute. Thus, after the assignments are done in this step, no state s repeats in any row or any column.

- (II) (a) For any row i , assign $s \in A_i$ to the entry (i, j) if A_i requires s in B_j to distribute after fixing $B_{\alpha(n_1)}, \dots, B_{\alpha(n_2)}$ where $\{(i, \alpha(n_j)) \mid 1 \leq j \leq r\}$ are the entries occupied by states assigned in the previous step.
 (b) For any column j , assign $s \in B_j$ to the entry (i, j) if B_j requires s in A_i to distribute after fixing $A_{\alpha(m_1)}, \dots, A_{\alpha(m_k)}$, where $\{(\alpha(m_i), j) \mid 1 \leq i \leq k\}$ are entries occupied by the previous step.

By the result of Remark 2.3, this process can be completed as for step I.

(III) Repeat step (II) until no new assignment is reached by such repetition.

(IV) Suppose for some row i , $\exists s \in A_i$ which has not been assigned by previous steps. Then, by Lemma 2.6 and Remark 2.3, there exists a column j $\ni s \in B_j$ which also has not been assigned. Assign s to this entry (i,j) . Then perform steps II, III again, until no new assignment is reached in III.

(V) Repeat step (IV) until every s in every coordinate A_i of has been assigned.

It is not hard to see that the result of these steps will give a rectangular table having no state s appearing repeatedly in any row or any column. Certainly, each s appears once in some row. But, it also appears once in some column. This completes the proof in one direction.

To prove the if part, we need only show that if such a table as described in the beginning of this proof exists, then (α, β) has DDP. Let E_α, E_β be defined as in Definition 2.11. For each s , if s appears more than once in the table, we can delete the state s in the i^{th} row and j^{th} column if s is in (i,j) . Then the resulting new pair of $*$ -covers is still MD. Since each state s appears exactly $m(s)$ times, we can perform this deletion for all $s \in S$ up to $(m(s)-1)$ times. Hence for each $s \in S$, $m(s) > 1$, (α, β) has the $(m(s)-1)^{\text{th}}$ - s -deleted distributive property. Therefore (α, β) has DDP. This completes the proof.

Q. E. D.

Using Theorem 2.2 as a stepping stone, we shall now establish a necessary and sufficient condition on a pair of $*$ -covers (α, β) of S to insure the existence of a function φ satisfying just (i), (ii) of Theorem 2.2 such that $\langle E_\alpha, E_\beta, \varphi \rangle$ is a GPDT of M .

First, we introduce the following definition.

Definition 2.13. Let α, β be $*$ -covers of S . Then (α, β) has the added deleted distributive property (ADDP) if and only if

- (i) There exist a finite set of symbols $\bar{S}$ and two $*$ -covers $\bar{\alpha}, \bar{\beta}$ of $\bar{S}$.
- (ii) There exists a function ρ from $\bar{S}$ onto S such that α is a $*$ -cover of S induced by the $*$ -cover $\bar{\alpha}$ of $\bar{S}$ under ρ , and β is a $*$ -cover of S induced by the $*$ -cover $\bar{\beta}$ of $\bar{S}$ under ρ .
- (iii) $(\bar{\alpha}, \bar{\beta})$ has DDP.

We shall show an example of $*$ -covers with ADDP. Let $S = \{s_1, s_2, \dots, s_5\}$. Let $\alpha = (\{s_1, s_2, s_3\}, \{s_1, s_3\}, \{s_4, s_5\})$, $\beta = (\{s_1, s_4\}, \{s_1, s_2, s_4\}, \{s_3, s_5\})$. Then there exists a finite set of symbols $\bar{S} = \{s_1', s_2', \dots, s_6'\}$. Let $\bar{\alpha} = (\{s_1', s_2', s_3'\}, \{s_1', s_7'\}, \{s_4', s_5', s_6'\})$, $\bar{\beta} = (\{s_1', s_4'\}, \{s_1', s_2', s_6'\}, \{s_3', s_5', s_7'\})$. Let $\rho: \bar{S} \rightarrow S$ such that $((\forall i)(1 \leq i \leq 5)(\rho(s_i') = s_i)) [\rho(s_6') = s_4] [\rho(s_7') = s_3]$. Then α, β are $*$ -covers of S with ADDP.

Theorem 2.3. Let $\alpha = (A_1, \dots, A_\ell)$, $\beta = (B_1, \dots, B_m)$ be $*$ -covers of S . Let E_α, E_β be defined as in Definition 2.11. Then (α, β) has the ADDP if and only if there exists a function φ on a subset E_β of $E_\alpha \times E_\beta$ onto S such that

- (i) For each coordinate A_i of α ,

$$A_i = \{ s \mid \exists j \in E_B \ni \varphi(i, j) = s \}$$

(ii) For each coordinate B_j of $\mathcal{B}$,

$$B_j = \{ s \mid \exists i \in E_A \ni \varphi(i, j) = s \}.$$

Proof. Since (α, β) has the ADDP, there exist (a) a finite set of symbols $\bar{S} = \{\bar{s}_i\}$, (b) two $*$ -covers $\bar{\alpha}, \bar{\beta}$ of $\bar{S}$, (c) a function ρ from $\bar{S}$ onto S such that α, β are $*$ -covers of S induced by $\bar{\alpha}, \bar{\beta}$ respectively under ρ . Furthermore, $(\bar{\alpha}, \bar{\beta})$ has DDP. Let $E_{\bar{\alpha}}, E_{\bar{\beta}}$ be defined as in Definition 2.11. By the result of Theorem 2.2, $(\bar{\alpha}, \bar{\beta})$ has DDP implies that there exists a function $\bar{\varphi}$ from a subset $E_{\bar{J}}$ of $E_{\bar{\alpha}} \times E_{\bar{\beta}}$ onto $\bar{S}$ such that

- (i) For each coordinate $\bar{A}_i = \{ \bar{s} \mid \exists j \in E_{\bar{\beta}} \ni \bar{\varphi}(i, j) = \bar{s} \}$.
- (ii) For each coordinate $\bar{B}_j = \{ \bar{s} \mid \exists i \in E_{\bar{\alpha}} \ni \bar{\varphi}(i, j) = \bar{s} \}$.
- (iii) For each $\bar{s} \in \bar{S}$, $\#\{(i, j) \mid \bar{\varphi}(i, j) = \bar{s}\} = \bar{m}(\bar{s})$, where $\bar{m}(\bar{s})$ is the multiplicity of $\bar{s}$ in $\bar{\alpha}$ or $\bar{\beta}$.

Since α, β are $*$ -covers induced by $\bar{\alpha}, \bar{\beta}$ respectively under ρ , $(\forall i)(1 \leq i \leq \ell)(A_i = \rho(\bar{A}_i))$, and $(\forall j)(1 \leq j \leq m)(B_j = \rho(\bar{B}_j))$.

Therefore $E_{\bar{\alpha}} = E_{\alpha}$, $E_{\bar{\beta}} = E_{\beta}$ and $E_{\bar{J}} = E_J$. Define a function $\varphi = \rho \bar{\varphi}$ from E_J onto S such that $\varphi(i, j) = s$ if $\rho(\bar{\varphi}(i, j)) = s$.

Thus, for each i , $A_i = \{ s \mid (\exists j \in E_B)(\varphi(i, j) = s) \}$, and for each j , $B_j = \{ s \mid (\exists i \in E_A)(\varphi(i, j) = s) \}$. This proves the only if part.

To show the if part, we note that the function φ gives rise to a $\ell \times m$ rectangular table in which there may be columns or rows containing repeated appearances of certain states. In this table, the rows are labeled by i , $1 \leq i \leq \ell$, and the columns are labeled by j , $1 \leq j \leq m$. We shall construct a new table where rows are again labeled by i , $1 \leq i \leq \ell$; columns are labeled by j , $1 \leq j \leq m$. For each occupied entry (i, j) , we associate a new

symbol, $\bar{s}(i,j)$. Let $\bar{S} = \{ \bar{s}(i,j) \mid (i,j) \text{ is occupied} \}$. For each (i,j) , let $\bar{s}(i,j)$ occupy the entry (i,j) in the new table.

Then we obtain a $\ell \times m$ table not necessarily all filled. For each

i , let $\bar{A}_i = \{ \bar{s}(i,j) \mid \bar{s}(i,j) \text{ occupies } (i,j) \}$. Similarly,

$\bar{B}_j = \{ \bar{s}(i,j) \mid \bar{s}(i,j) \text{ occupies } (i,j) \}$. Let $\bar{\mathcal{A}} = (\bar{A}_1, \dots, \bar{A}_\ell)$ and

$\bar{\mathcal{B}} = (\bar{B}_1, \dots, \bar{B}_m)$. Define a function ρ from $\bar{S}$ onto S as follows.

For each $\bar{s}(i,j) \in \bar{S}$, $\rho(\bar{s}(i,j)) = s$ if s occupies (i,j) in the

original table. Certainly $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ are $*$ -covers of $\bar{S}$. And for

each i , $1 \leq i \leq \ell$, $\rho(\bar{A}_i) = \{ \rho(\bar{s}(i,j)) \mid \bar{s}(i,j) \in \bar{A}_i \}$

$$= \{ s \mid s \text{ is in the } i^{\text{th}} \text{ row} \}$$

$$= A_i.$$

Similarly, for each j , $1 \leq j \leq m$, $\rho(\bar{B}_j) = B_j$. Therefore, $\mathcal{A}, \mathcal{B}$

are $*$ -covers of S induced by the $*$ -covers $\bar{\mathcal{A}}, \bar{\mathcal{B}}$ of $\bar{S}$ respec-

tively under ρ . It is obvious that $(\bar{\mathcal{A}}, \bar{\mathcal{B}})$ has DDP. There-

fore, $(\mathcal{A}, \mathcal{B})$ has ADDP. This completes the proof.

Q.E.D.

We have now reached the following main theorem.

Theorem 2.4. All GPDT of $M = \langle S, T, f \rangle$ are of the type

$\langle E_{\mathcal{A}}, E_{\mathcal{B}}, \varphi \rangle$, where $\mathcal{A}, \mathcal{B}$ are $*$ -covers of S , and $(\mathcal{A}, \mathcal{B})$

has ADDP.

Chapter II

PROPERTIES OF GENERALIZED PAIR DECOMPOSITION

At the beginning of this chapter, we shall introduce the notion of a monadic algebra and its relation to an automaton. Our definition for an algebra (3) will be a rather general one. This allows us to state many theorems with appropriate degree of generality.

By an algebra Ω , we shall mean a triplet $\langle S, T, O = \{f_\alpha\} \rangle$, where T is a set of symbols called operation indices, S is a set called the carrier of elements, and O is a set of single-valued functions, $O = \{f_\alpha \mid \alpha \in T \ni f_\alpha : S^n \rightarrow S \text{ and } n = n(\alpha)\}$. If $n(\alpha) = 1$ for all $\alpha \in T$, then Ω is called a monadic algebra.

Let $M = \langle S, T, f \rangle$ be a deterministic complete automaton. Then M is a monadic algebra in the following way: Let T be the set of operation indices and S be the carrier set. For each $\alpha \in T$ define an operator $f_\alpha : S \rightarrow S$ such that $(\forall s)[f_\alpha(s) = f(s, \alpha)]$. Let $O = \{f_\alpha \mid \alpha \in T\}$, then $\langle S, T, O \rangle$ is clearly a monadic algebra. Conversely, any abstract monadic algebra $\langle S, T, O = \{f_\alpha\} \rangle$ can be considered as a deterministic complete automaton. Let S be the set of states, T the set called the input alphabet. The transition function f is defined such that $(\forall (s, \alpha) \in S \times T)[f(s, \alpha) = f_\alpha(s)]$. As a consequence, any result of monadic algebra will hold for an automaton.

We shall now introduce the notion of an incomplete monadic algebra. In an incomplete monadic algebra $\langle S, T, O \rangle$, each operator f_α is defined on a subset of S . Certainly a

monadic algebra is also an incomplete monadic algebra. If we follow the analogous formulation as discussed before, we can show that an incomplete monadic algebra is an incomplete deterministic automaton and vice versa. Therefore, in the following, we make no distinction between an automaton and monadic algebra.

§1. Algebraic Properties of Congruence Relations of Degree k .

Let $M = \langle S, T, f \rangle$ be an incomplete automaton. Let $\{M_1 : M_2\}$ be a given GPD of M , where $M_1 = \langle S_1, T_1, f_1 \rangle$ and $M_2 = \langle S_2, T_2, f_2 \rangle$ are incomplete automata. By the result of Chapter I, M is thus a homomorphic image of a sub-automaton $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ of the generalized composition of M_1, M_2 . (See definition on page 8.) We will call $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ the induced automaton of the GPD $\{M_1 : M_2\}$.

In fact, $\{M_1 : M_2\}$ is also a GPD of $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$, and the automaton M as related to the GPD $\{M_1 : M_2\}$ is completely represented by $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$. Thus, in order to find properties of a GPD $\{M_1 : M_2\}$ of M , we must study $\{M_1 : M_2\}$ as a GPD of the induced automaton $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ or any isomorphic image $\langle S', T, f' \rangle$ of $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$. If $M' = \langle S', T, f' \rangle$ is isomorphic to $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ where σ is the isomorphism, then $\{M_1 : M_2\}$ is also a GPD of $\langle S', T, f' \rangle$. Furthermore, if $\langle S_1, S_2, \psi \rangle$ is the associated GPDT of $\{M_1 : M_2\}$ for M' , then ψ is the isomorphism σ from $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ onto $\langle S', T, f' \rangle$. Thus, we need only study the sort of GPD $\{M_1 : M_2\}$ of an automaton M where in its associated GPDT $\langle S_1, S_2, \varphi \rangle$ the homomorphism φ is an isomorphism.

Let $\{M_1 : M_2\}$ be a GPD of $M = \langle S, T, f \rangle$. Let the associated GPDT be $\langle S_1, S_2, \varphi \rangle$ where φ is one to one from $\mathcal{S}$ onto S .

Define a relation θ_{11} on S such that $(\forall s, s' \in S) [(s\theta_{11}s') \iff [\pi_1(\varphi^{-1}(s)) = \pi_1(\varphi^{-1}(s'))]]$. Similarly, we define a relation θ_{22} . It can be easily shown that θ_{11} and θ_{22} are equivalence relations. Further, in the lattice of all equivalence relations on S , the g.l.b. of θ_{11} and $\theta_{22} = \theta_{11} \cap \theta_{22} = 0$. Since if $s \neq s'$ and $s(\theta_{11} \cap \theta_{22})s'$, then $\pi_i[\varphi^{-1}(s)] = \pi_i[\varphi^{-1}(s')]$ for $1 \leq i \leq 2$, therefore $\varphi^{-1}(s) = \varphi^{-1}(s')$. Yet φ is one to one onto, therefore it leads to a contradiction. Hence, automata M_1, M_2 give rise to two equivalence relations θ_{11}, θ_{22} on S (thus two partitions on S) such that $\theta_{11} \cap \theta_{22} = 0$.

We shall now study properties of certain equivalence relations on a monadic algebra which are pertinent to the theory of decomposition of automata.

Let $\Omega = \langle S, T, 0 \rangle$ be an incomplete monadic algebra. Let $T = \{\alpha_1, \dots, \alpha_\ell\}$. Let θ be an equivalence relation on S . For every $s \in S$, $f(s, T)$ denotes the ℓ -tuple $(s_1, \dots, s_\ell)$ where $(\forall i)[s_i = f_{\alpha_i}(s)]$. On the set of all ℓ -tuples of states of S , θ induces an equivalence relation $\tilde{\theta}$ such that

$$[(s_1, \dots, s_\ell)\tilde{\theta}(s_1', \dots, s_\ell') \iff (\forall i)[s_i \theta s_i']].$$

Definition 1.1. Given an incomplete monadic algebra

$\Omega = \langle S, T, \{f_\alpha\} \rangle$, θ is a congruence relation of degree k if and only if

(i) θ is an equivalence relation on S .

(ii) Let $N(s, \theta)$ be the number of distinct equivalence classes under $\tilde{\theta}$ having a non-empty intersection with $\{f(s', T) \mid s' \theta s\}$. Then $k = k(\theta) = \max_{s \in S} N(s, \theta)$.

We note that a congruence relation of degree 1 is the ordinary congruence relation (i.e., an equivalence relation

satisfying the substitution property (3)).

Proposition 1.1. Let θ_1, θ_2 be congruence relations on of degree k_1, k_2 respectively. Then $(\theta_1 \cap \theta_2)$ is a congruence relation of degree $\leq \min(k_1 k_2, \#(R))$ where $\#(R)$ is the number of distinct equivalence classes under θ_3 in the range space $R = \{f(s, T) \mid s \in S\}$ where $\theta_3 = (\theta_1 \cap \theta_2)$.

Proof. It is obvious that $(\theta_1 \cap \theta_2)$ is an equivalence relation on S . It also can be easily shown that the number of equivalence classes of $(\theta_1 \cap \theta_2)$ in the set $\{f(s', T) \mid s' (\theta_1 \cap \theta_2) s\}$ for any $s \in S$ is $\leq k_1 k_2$. However, the number of equivalence classes of $(\theta_1 \cap \theta_2)$ is also restricted by the number of equivalence classes in the range space. Therefore, $(\theta_1 \cap \theta_2)$ is a congruence relation of degree $\leq \min(\#(R), k_1 k_2)$.

Q. E. D.

The following example will show that if θ_1, θ_2 are congruence relations of degree k_1, k_2 respectively, then the l.u.b. of θ_1 and $\theta_2 = (\theta_1 \cup \theta_2)$ may be of degree higher than $k_1 k_2$.

Let $M = \langle S, T, f \rangle$. Let $S = \{1, 2, \dots, 18\}$, $T = \{\alpha\}$.

The transition table is:

f	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
α	10	11	12	13	14	15	16	17	18	10	11	12	13	14	15	16	17	18

Let $\theta_1 = (\{123\}, \{456\}, \{789\}, \{10\}, \{11\}, \{12\}, \dots, \{18\})$, then θ_1 is of degree 3. Let $\theta_2 = (\{12\}, \{34\}, \{5\}, \{67\}, \{89\}, \{10\}, \dots, \{18\})$, then θ_2 is of degree 2. Then $(\theta_1 \cup \theta_2) = (\{123456789\}, \{10\}, \{11, \dots, 18\})$ is of degree $9 > k_1 k_2 = 6$.

Definition 1.2. Let $\Omega = \langle S, T, \{f_\alpha\} \rangle$ be an incomplete monadic algebra. Let θ be an equivalence relation on S . Then

$f^{-1}(\theta)$ is a relation defined as follows:

$$(\forall s, s' \in S) \left[(sf^{-1}(\theta)s') \iff (\forall \alpha) [f_\alpha(s) \theta f_\alpha(s')] \right].$$

In general, $f^{-1}(\theta)$ may not be an equivalence relation. However, if $\Omega = \langle S, T, \{f_\alpha\} \rangle$ is complete, then $f^{-1}(\theta)$ is also an equivalence relation.

Definition 1.3. Let θ_1, θ_2 be two relations on S , then

$$(\theta_1 \supseteq \theta_2) \text{ if and only if } (\forall s, s' \in S) [(s\theta_2 s') \Rightarrow (s\theta_1 s')].$$

Definition 1.4. Let θ_1, θ_2 be equivalence relations on S such that $\theta_1 \supseteq \theta_2$. Then (θ_1/θ_2) is an equivalence relation on the set $\mathcal{E}$, where $\mathcal{E}$ is the family of equivalence classes of S under θ_2 . For each $s \in S$, let $\bar{s} \in \mathcal{E}$. Then we define:

$$[\bar{s}_1(\theta_1/\theta_2)\bar{s}_2] \iff (\forall s_1')(\forall s_2') \left[[(\bar{s}_1 = \bar{s}_1')(\bar{s}_2 = \bar{s}_2')] \Rightarrow (s_1' \theta_1 s_2') \right].$$

For any equivalence relation θ on S , let $m(\theta)$ denote the maximum number of elements in any class of θ and $n(\theta)$ the number of equivalence classes of θ .

Proposition 1.2. Given a complete monadic algebra $\Omega = \langle S, T, \{f_\alpha\} \rangle$, then θ is a congruence relation of degree k if and only if

- (i) θ is an equivalence relation on S
- (ii) $m\left(\frac{\theta}{f^{-1}(\theta) \cap \theta}\right) = k.$

Proof. We need only show that condition (ii) here is the same as condition (ii) in Definition 1.1 when Ω is a monadic algebra. This is easily done. Since Ω is complete, $f^{-1}(\theta)$ is also an equivalence relation on S . Therefore, $m\left(\frac{\theta}{f^{-1}(\theta) \cap \theta}\right)$ is the maximum number of distinct equivalence classes under θ on the set $\{f(s', T) \mid s' \theta s\}$, for any $s \in S$.

Q. E. D.

Proposition 1.3. Let θ_1, θ_2 be congruence relations of degree k_1, k_2 respectively such that $\theta_1 \supseteq \theta_2$. Then $k_1 \leq k_2$ if, for each fixed $s \in S$, for every $s' \in S$, $(s' \theta_1 s)$, $\exists t \in S$, $t \theta_2 s$, $\ni (\forall \alpha \in T) [f_\alpha(s') \theta_1 f_\alpha(t)]$.

Proof. If for each s' , $(s' \theta_1 s)$, $\exists t \in S$, $t \theta_2 s$, $\ni (\forall \alpha) [f_\alpha(s') \theta_1 f_\alpha(t)]$, then, in particular, this is true for state s' , $s' \theta_1 s$, yet $s' \not\theta_2 s$. Thus, all such elements s' are not mapped by f to any addition equivalence classes of θ_2 . Hence $k_1 \leq k_2$.

Q.E.D.

We shall show by an example that the restriction $\theta_2 \subseteq \theta_1$ in the above proposition can not be lessened. Let $\Omega = \langle S, T, f_\alpha \rangle$ be given as follows:

f	0	1	2	3	4	5	6	7
α	7	7	5	6	7	5	7	7

Let $\theta_1 = \{\{0, 1\}, \{2, 3, 4\}, \{5, 6\}, \{7\}\}$, $\theta_2 = \{\{0\}, \{1, 2, 3, 4\}, \{5\}, \{6\}, \{7\}\}$.

Although all other conditions are satisfied except $\theta_2 \not\subseteq \theta_1$, we have that the degree of θ_2 is 3 and the degree of θ_1 is 2.

Definition 1.5. Let θ_{11}, θ_{22} be equivalence relations on S

such that $\theta_{11} \cap \theta_{22} = 0$. Then θ_{21} is a symmetric, reflexive relation on S defined as follows. For $s_1, s_2 \in S$,

$(s_1 \theta_{21} s_2)$ if and only if

$$(\forall \alpha)(\forall s_1')(\forall s_2') \left[((s_1' \theta_{22} s_1)(s_2' \theta_{22} s_2)(s_1' \theta_{11} s_2')) \right. \\ \left. \implies [f(s_1', \alpha) \theta_{11} f(s_2', \alpha)] \right].$$

Proposition 1.4.

- (i) $\theta_{12} \supseteq \theta_{11}$ and $\theta_{21} \supseteq \theta_{22}$
- (ii) $\theta_{12} \subseteq f^{-1}(\theta_{22})$ and $\theta_{21} \subseteq f^{-1}(\theta_{11})$.

Proof. Part (i) is easily shown by noting that $\{\theta_{ii}\}_2$ are equivalence relations such that $\bigcap_{i=1}^2 \theta_{ii} = 0$. To prove part (ii),

let $s_1 \theta_{12} s_2$. Then for each s_1' and for each s_2' such that $(s_1' \theta_{11} s_1)$ and $(s_2' \theta_{11} s_2)$ and $(s_1' \theta_{22} s_2')$ we have that $(\forall \alpha)[f_\alpha(s_1') \theta_{22} f_\alpha(s_2')]$. In particular, $(\forall \alpha)[f_\alpha(s_1) \theta_{22} f_\alpha(s_2)]$.

Q.E.D.

With the result of the above proposition and the discussion on pages 28, 29 of this chapter, we have the following theorem.

Theorem 1.1. Let $[M_1:M_2]$ be a GPD of $M = \langle S, T, f \rangle$. Let the associated GPDT be $\langle S_1, S_2, \rho \rangle$ such that ρ is one to one from $\mathcal{S} \subseteq S_1 \times S_2$ onto S and $\pi_1(\mathcal{S}) = S_1$, $\pi_2(\mathcal{S}) = S_2$. Then $[M_1:M_2]$ gives rise to a set of relations $\{\theta_{ij} | 1 \leq i, j \leq 2\}$ on S such that

- (i) θ_{11}, θ_{22} are equivalence relations on S and $\theta_{11} \cap \theta_{22} = 0$.
- (ii) $\theta_{12} \supseteq \theta_{11}$
 $\theta_{21} \supseteq \theta_{22}$
- (iii) $\theta_{12} \subseteq f^{-1}(\theta_{22})$
 $\theta_{21} \subseteq f^{-1}(\theta_{11})$

For the result in the other direction, we have the following theorem.

Theorem 1.2. Let $M = \langle S, T, f \rangle$. Let $\{\theta_{ij} | 1 \leq i, j \leq 2\}$ be a set of relations on S such that

- (i) θ_{11}, θ_{22} are equivalence relations on S and $\theta_{11} \cap \theta_{22} = 0$.
- (ii) $\theta_{12} \supseteq \theta_{11}$
 $\theta_{21} \supseteq \theta_{22}$
- (iii) $\theta_{12} \subseteq f^{-1}(\theta_{22})$
 $\theta_{21} \subseteq f^{-1}(\theta_{11})$.

Then there exist two incomplete automata $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ such that

- (i) $\{M_1: M_2\}$ is a GPD of M
- (ii) φ is $1 \rightarrow 1$ where $\langle S_1, S_2, \varphi \rangle$ is its associated GPDT.
- (iii) $\{M_1: M_2\}$ gives rise to the set $\{\theta_{ij} \mid 1 \leq i, j \leq 2\}$.

Proof. Let $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ be defined as follows. Let S_1 be the family of equivalence classes under θ_{11} . Similarly, let S_2 be the family of equivalence classes under θ_{22} . For each $s \in S$, let $(s)_1 \in S_1$ and let $(s)_2 \in S_2$. Let $T_1 = S_2 \times T$, $T_2 = S_1 \times T$. Define $f_1: S_1 \times T_1 \rightarrow S_1$ such that for all $((s)_1, (s')_2, \alpha)$ if $(s)_1 \cap (s')_2 \neq \emptyset$, then $f_1((s)_1, (s')_2, \alpha) = f((s)_1 \cap (s')_2, \alpha)$. This is well-defined. Since $\theta_{11} \cap \theta_{22} = 0$, if $(s)_1 \cap (s')_2 \neq \emptyset$, then there exists exactly one unique element in the intersection. We define f_2 similarly.

We now proceed to show that $\{M_1: M_2\}$ is a GPD of M . Define $\mathcal{S} \subseteq S_1 \times S_2$ such that $\mathcal{S} = \{((s)_1, (s')_2) \mid (s)_1 \cap (s')_2 \neq \emptyset\}$. Let φ be a function from $\mathcal{S}$ onto S such that $\{((s)_1, (s')_2) \in \mathcal{S} \mid \varphi((s)_1, (s')_2) = (s)_1 \cap (s')_2\}$. Clearly, $\{M_1: M_2\}$ is a GPD of M and if $\langle S_1, S_2, \varphi \rangle$ is its associated GPDT, φ is $1 \rightarrow 1$.

By the construction of M_1, M_2 , it is obvious that M_1, M_2 induces two equivalence relations θ_{11}, θ_{22} respectively and $\theta_{11} \cap \theta_{22} = 0$. Let $s_1, s_2 \in S$. Let $(s_1 \theta_{12} s_2)$. Let $s_1', s_2' \in S$ such that $(s_1' \theta_{11} s_1)(s_2' \theta_{11} s_2)(s_1' \theta_{22} s_2')$. Then since $\theta_{12} \subseteq f^{-1}(\theta_{22})$, $(\forall \alpha \in T)[f(s_1, \alpha) \theta_{22} f(s_2, \alpha)]$. Also, $\theta_{11} \subseteq \theta_{12} \subseteq f^{-1}(\theta_{22})$, thus $(\forall \alpha)[f(s_1', \alpha) \theta_{22} f(s_1, \alpha)]$ and $(\forall \alpha)[f(s_2', \alpha) \theta_{22} f(s_2, \alpha)]$. Since θ_{22} is transitive, $(\forall \alpha)[f(s_1', \alpha) \theta_{22} f(s_2', \alpha)]$. Hence, if $(s_1 \theta_{12} s_2)$ then

$$(\forall \alpha)(\forall s_1')(\forall s_2') \left\{ [(s_1' \theta_{11} s_1)(s_2' \theta_{11} s_2)(s_1' \theta_{22} s_2)] \right. \\ \left. \implies [f(s_1', \alpha) \theta_{22} f(s_2', \alpha)] \right\}.$$

We can construct a similar proof for θ_{21} . Thus, $[M_1:M_2]$ gives rise to the set $\{\theta_{ij} | 1 \leq i, j \leq 2\}$.

Q.E.D.

We define a cover $\mathcal{C}$ of a given set A as a collection of non-empty subsets A_k of A such that their union is A . Any subcollection of elements in $\mathcal{C}$ whose union is A is a subcover of $\mathcal{C}$. A minimum subcover $\mathcal{C}_M$ is a subcover of a given cover $\mathcal{C}$ such that it has the least cardinality. The classical covering problem is to find a minimum subcover $\mathcal{C}_M$ of a given cover $\mathcal{C}$.

For any reflexive, symmetric relation β on the set S , we can represent the relation β by a cover $\mathcal{C}_\beta$ of S . Each subset of S in this cover $\mathcal{C}_\beta$ is a collection of elements which mutually bear the β relation. Thus, if we consider θ_{ij} , $i \neq j$ in particular, we can also represent this relation by a cover of S . In the case when the relation β is an equivalence relation, the cover is represented by the collection of its equivalence classes. In this particular case, the cover is itself the minimum subcover. We shall let $n(\beta)$ denote the number of elements in a minimum subcover of the cover $\mathcal{C}_\beta$, where β is a symmetric and reflexive relation on S . This number $n(\beta)$ is unique.

§2. Congruence Relations of Degree k as Applied to Decomposition Theory

In §1, we have developed properties of congruence relations of degree k . In this section, we consider its application to the theory of decomposition. Again, we shall study

GPD $\{M_1:M_2\}$ of $M = \langle S, T, f \rangle$ where φ is a one to one function from $\mathcal{S} \subseteq S_1 \times S_2$ onto $S \ni \pi_1(\mathcal{S}) = S_1, \pi_2(\mathcal{S}) = S_2$. In this case, M is an isomorphic image of a sub-automaton $\langle \mathcal{S}, T, F|_{\mathcal{S}} \rangle$ of $G(M_1, M_2)$. Automata M_1, M_2 induce two equivalence relations θ_{11}, θ_{22} on S , $\theta_{11} \cap \theta_{22} = 0$.

Let $S_1 = \{a_1, \dots, a_{n_1}\}$, $S_2 = \{b_1, \dots, b_{n_2}\}$. For each $a_i \in S_1$, define $f_{a_i}: S_2 \times T$ into S_2 such that $(\forall (b_j, \alpha) \in S_2 \times T) [f_{a_i}(b_j, \alpha) = \pi_2[\varphi^{-1}\{f(\varphi(a_i, b_j), \alpha)\}]]$. Similarly, $(\forall (a_i, \alpha) \in S_1 \times T) [f_{b_j}(a_i, \alpha) = \pi_1[\varphi^{-1}\{f(\varphi(a_i, b_j), \alpha)\}]]$. We note that for each a_i , $D(f_{a_i}) = \{(b_j, \alpha) \mid \alpha \in T, (a_i, b_j) \in D(\varphi) \text{ and } (\varphi(a_i, b_j), \alpha) \in D(f)\}$, and for each b_j , $D(f_{b_j}) = \{(a_i, \alpha) \mid \alpha \in T, (a_i, b_j) \in D(\varphi) \text{ and } (\varphi(a_i, b_j), \alpha) \in D(f)\}$.

Definition 2.1. For any $a_1, a_2 \in S_1$, f_{a_1} is compatible with f_{a_2} ; $f_{a_1} \sim f_{a_2}$ if and only if they are the same mapping over their common domain (i.e., $D(f_{a_1}) \cap D(f_{a_2})$).

Similarly, we define for b_1, b_2 , $f_{b_1} \sim f_{b_2}$.

We note that compatibility is not an equivalence relation. It is an equivalence relation when $n_1 n_2 = \#(S)$. In that case, the common domain for all f_a is $S_2 \times T$, and for all f_b is $S_1 \times T$.

Suppose we consider a set of mutually compatible functions f_{a_i} as one function. As a_i ranges over S_1 , we obtain a collection of such distinct non-compatible functions. Let p be the number of elements in this collection. Since compatibility is not transitive, as a_i ranges over S_1 we may obtain several collections of such distinct functions. We denote the minimum p by F_{S_1} as we range over all possible

collections. Similarly we use F_{S_2} . In view of the nature of these functions we shall call F_{S_1} the number of input configurations from M_1 to M_2 , and F_{S_2} the number of input configurations from M_2 to M_1 .

For each such collection of non-compatible functions from M_1 to M_2 and from M_2 to M_1 , we must encode it into alphabetical information from M_1 to M_2 and from M_2 to M_1 respectively. These, in turn, must be coded into binary signals. If we denote by $\lceil x \rceil$ the least non-negative integer $\geq x$ for any real number x , then we need $\lceil \log_2 F_{S_1} \rceil$ and $\lceil \log_2 F_{S_2} \rceil$ numbers of binary signals. We shall call $\lceil \log_2 F_{S_1} \rceil$, $\lceil \log_2 F_{S_2} \rceil$ the number of connecting lines from M_1 to M_2 and M_2 to M_1 respectively.

We have so far introduced the notions of input configurations and connecting lines between machines. The next natural question is how does one find these numbers? First, we show the following proposition.

Proposition 2.1. Let $s_1, s_2 \in S \ni \pi_1(\bar{\varphi}^1(s_1)) = a_1$,

$\pi_1(\bar{\varphi}^1(s_2)) = a_2$. Then $f_{a_1} \sim f_{a_2}$ if and only if $s_1 \theta_{12} s_2$.

Proof. The statement really says that Definition 1.5 and

Definition 2.1 are equivalent. This is quite straightforward.

Since θ_{11} is defined such that $(s_i \theta_{11} s_j)$ if and only if

$\pi_1(\bar{\varphi}^1(s_i)) = \pi_1(\bar{\varphi}^1(s_j))$, and θ_{22} is defined such that $(s_i \theta_{22} s_j) \Leftrightarrow [\pi_2(\bar{\varphi}^1(s_i)) = \pi_2(\bar{\varphi}^1(s_j))]$, $f_{a_1} \sim f_{a_2}$ really means $s_1 \theta_{12} s_2$, and vice versa.

Q. E. D.

Similarly, we have a proposition for f_b . Let

$s_1, s_2 \in S \ni \pi_2(\bar{\varphi}^1(s_1)) = b_1$, $\pi_2(\bar{\varphi}^1(s_2)) = b_2$. Then $f_{b_1} \sim f_{b_2}$ if and only if $s_1 \theta_{21} s_2$.

As a result of the above proposition and the discussion

at the end of $\mathcal{S}_1$, the problem of finding the numbers of input configurations becomes a covering problem [19]. This covering problem is the same problem as, for example, the state minimization of incomplete machines. Thus a detailed study of this type of problem can be found in several articles [2,13,16,18]. Using the notation in $\mathcal{S}_1$, we denote by $n(\theta_{12})$ and $n(\theta_{21})$ the numbers of input configurations from M_1 to M_2 and M_2 to M_1 respectively.

Let $M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ such that $\langle M_1 : M_2 \rangle$ is a GPD of M . Let $\langle S_1, S_2, \varphi \rangle$ be the associated GPDT, and let φ be one to one. For each $a \in S_1$, each $b \in S_2$, there exists an ℓ -tuple $(a_1, \dots, a_\ell)$ such that (i) For each i , $(a, \alpha_i) \in D(f_b)$, then $a_i = \{f_b(a, \alpha_i)\}$ (ii) For each i , $(a, \alpha_i) \notin D(f_b)$, then $a_i = \emptyset$. For each $a \in S_1$, let the collection of all such ℓ -tuples as b ranges over S_2 be the successor set of a . Define two ℓ -tuples $(a_1, \dots, a_\ell)$ and $(a_1', \dots, a_\ell')$ as equal if and only if for each i where $a_i \neq \emptyset$ and $a_i' \neq \emptyset$, then $a_i = a_i'$. Let L_{S_1} denote the maximum number of distinct ℓ -tuples in any successor set of a as a ranges over S_1 .

Proposition 2.2. The number L_{S_1} is a lower bound on the number of distinct non-compatible functions (input configurations) from M_2 to M_1 .

Proof. This is a direct result of the definition of L_{S_1} . There are at least L_{S_1} distinct f_{b_j} , since these f_{b_j} map the same input-state pair to distinct next state. Therefore L_{S_1} is a lower bound.

Q. E. D.

Recall the definition of a congruence relation θ of degree k . The number $N(s, \theta_{11})$ is thus defined as the number

of distinct equivalence classes of $\tilde{\theta}_{11}$ in $\{f(s',T) \mid s' \theta_{11} s\}$. The maximum of $N(s, \theta_{11})$ for any s is $\kappa = \kappa(\theta_{11})$ and is exactly the number L_{S_1} . Therefore, we have the following proposition.

Proposition 2.3. There exists a congruence relation θ_{11} on S of degree k if and only if there exists a GPD $[M_1:M_2]$ of M such that

- (i) M_1 induces θ_{11} .
- (ii) φ is $1 \rightarrow 1$ where $\langle S_1, S_2, \varphi \rangle$ is the associated GPDT.
- (iii) k is a lower bound on the number of input configurations from M_2 to M_1 .

Let θ_{11} be a congruence relation of degree k , $k > 1$, on S . Suppose θ_{22} is any equivalence relation on S such that $\theta_{11} \cap \theta_{22} = 0$. Then $n(\theta_{21})$ is not necessarily equal to k , i.e., k is only a lower bound on the number of input configurations from M_2 to M_1 . This can be easily explained by an example. Let $M = \langle S, T, f \rangle$ where $S = \{1, 2, \dots, 6\}$ and $T = \{\alpha\}$. Let the transition function be:

f	1	2	3	4	5	6
α	3	4	5	6	2	4

Let $\theta_{11} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, then θ_{11} is of degree 2. Let $S_1 = \{a_1, a_2\}$ where a_1 denotes $\{1, 2, 3\}$, and a_2 denotes $\{4, 5, 6\}$. Suppose we choose $\theta_{22} = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Let $S_2 = \{b_1, b_2, b_3\}$ where b_1 denotes $\{1, 4\}$, b_2 denotes $\{2, 5\}$, b_3 denotes $\{3, 6\}$. It can easily be shown that $n(\theta_{21}) = 3$.

Our next result will show that if $k = 1$, k is the number of input configurations from M_2 to M_1 .

Proposition 2.4. There exists an ordinary congruence relation θ_{11} on S if and only if there exist two incomplete automata

$M_1 = \langle S_1, T_1, f_1 \rangle$, $M_2 = \langle S_2, T_2, f_2 \rangle$ such that $[M_1:M_2]$ is a GPD of M and

- (i) M_1 induces θ_{11} .
- (ii) φ is $1 \rightarrow 1$ where $\langle S_1, S_2, \varphi \rangle$ is the associated GPDT.
- (iii) The number of input configurations from M_2 to M_1 is 1.

Proof. (i), (ii) can be obtained directly from Proposition 2.3. To show (iii), we need only show that $(\forall s_1, s_2 \in S)(s_1 \theta_{21} s_2)$, i.e., $n(\theta_{21}) = 1$. This is quite simple. Let $s_1, s_2 \in S$. Let $(s_1' \theta_{22} s_1)(s_2' \theta_{22} s_2)(s_1' \theta_{11} s_2')$. Since θ_{11} is an ordinary congruence relation, $(s_1' \theta_{11} s_2') \Rightarrow (\forall \alpha)[f(s_1', \alpha) \theta_{11} f(s_2', \alpha)]$. Thus $(\forall s_1, s_2 \in S)[s_1 \theta_{21} s_2]$.

Q.E.D.

Remark 2.1. We shall show in Chapter III that the result of this special case includes the known results [12] about series decomposition of an automaton.

If we combine the results of Theorem 1.2 and Proposition 2.3, we reach the following.

Proposition 2.5. There exist two congruence relations θ_{11} , θ_{22} of degree k_1 , k_2 respectively $\ni \theta_{11} \cap \theta_{22} = 0$ if and only if there exists a GPD $[M_1:M_2]$ of M

- (i) M_1 , M_2 induce θ_{11} , θ_{22} respectively.
- (ii) φ is $1 \rightarrow 1$ where $\langle S_1, S_2, \varphi \rangle$ is the associated GPDT.
- (iii) k_1 , k_2 are the lower bounds of the numbers of input configurations from M_2 to M_1 and M_1 to M_2 respectively.

Finally, with the aid of proposition 2.4, we have

Proposition 2.6. There exist two ordinary congruence relations θ_{11} , θ_{22} on S such that $\theta_{11} \cap \theta_{22} = 0$ if and only if there exists a GPD $[M_1:M_2]$ of M such that

- (i) M_1, M_2 induce Θ_{11}, Θ_{22} respectively.
- (ii) φ is $1 \rightarrow 1$ where $\langle S_1, S_2, \varphi \rangle$ is its associated GPDT.
- (iii) The numbers of input configurations from M_2 to M_1 and from M_1 to M_2 are 1.

Remark 2.2. We shall show in Chapter III that the result of this special case includes the known results [12] about parallel decomposition of an automaton.

We have shown that if Θ_{11} is a congruence relation of degree k on S and if $[M_1:M_2]$ is a GPD of $M = \langle S, T, f \rangle$ and the states of M_1 induce Θ_{11} on S , then in general K is just a lower bound on the number of input configurations from M_2 to M_1 . We shall now establish the result that this lower bound k can always be reached if we allow M_2 to vary. We shall obtain an upper bound on the number of states of such M_2 and show that it is always less than the number of states in the original automaton. Finally, we show by an example that, in fact, this upper bound is sometimes the least upper bound.

Let $M = \langle S, T, f \rangle$ and $\# \{S\} = n$. Let Θ_{11} be a given congruence relation of degree k . Denote by $\{P_i \mid 1 \leq i \leq n(\Theta_{11})\}$ the set of equivalence classes of Θ_{11} . Let $\eta(P_i)$ be the number of distinct $f^{-1}(\Theta_{11})$ classes in P_i . Let ℓ_i denote the number of elements in each equivalence class P_i . We shall choose P_1 to be a particular equivalence class of Θ_{11} such that $\eta(P_1) = k$ and if, for some j , $\eta(P_j) = j$, then $\ell_j \leq \ell_1$.

Definition 2.2. For each i , $1 \leq i \leq n(\Theta_{11})$, define

$$g(i) = \begin{cases} \ell_i - (k+1) & \text{if } \ell_i > k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.1. Given a congruence relation Θ_{11} of degree k , there exists an equivalence relation Θ_{22} on S which, together with Θ_{11} ,

gives rise to a reflexive, symmetric relation Θ_{21} as defined in Definition 1.5, such that

- (i) $\Theta_{11} \cap \Theta_{22} = 0$.
- (ii) $n(\Theta_{22}) = \begin{cases} k & \text{if } (\forall i)(\ell_i \leq k) \\ (k+1) + \sum_{i=1}^{n(\Theta_{11})} g(i) & \text{otherwise} \end{cases}$
- (iii) $m(\Theta_{11}) \leq n(\Theta_{22}) \leq n$
and $n(\Theta_{22}) = n$ only if $n(\Theta_{11}) = 1$.
- (iv) $\Theta_{21} \supseteq \Theta_{22}$
- (v) $n(\Theta_{21}) = k$.

Proof. For each P_i , order the elements in P_i such that the first $\eta(P_i)$ states give a complete set of representatives for the $f^{-1}(\Theta_{11})$ equivalence classes in P_i . For each i , let $s_{i,t}$ denote the t^{th} state in P_i .

Case I. $(\forall i)(\ell_i \leq k)$.

Let $Q_1, \dots, Q_k$ be k subsets of S defined as follows:

For each i , $1 \leq i \leq n(\Theta_{11})$

(1) For any t , $1 \leq t \leq \ell_i$, $s_{i,t} \in Q_j$ for one and only one j , $1 \leq j \leq k$.

(2) For any t , $t' \ni t \neq t'$ and $1 \leq t, t' \leq \ell_i$, and if $(s_{i,t} \in Q_j)$ and $(s_{i,t'} \in Q_{j'})$, then $j \neq j'$.

Clearly, by (1), $\{Q_i\}_{1 \leq i \leq k}$ is a collection of disjoint subsets of S . By (2), no two distinct states in any Θ_{11} -equivalence class belong to the same Q_i . Since at least $\ell_1 = k$, we have, for any j , $1 \leq j \leq k$, Q_j is non-empty. Clearly, $\bigcup_{i=1}^k Q_i = S$. Further, $\{Q_i\}$ forms a partition on S ; thus induces an equivalence relation Θ_{22} . Therefore, we have $\Theta_{11} \cap \Theta_{22} = 0$ and $n(\Theta_{22}) = k$. From Θ_{11} , Θ_{22} , define Θ_{21} as in Definition 1.5. Hence $\Theta_{21} \supseteq \Theta_{22}$, by Theorem 1.1. Thus $n(\Theta_{22}) = k \geq n(\Theta_{21}) \geq k$. Thus $n(\Theta_{21}) = k$.

Case II. $(\exists i)(1 \leq i \leq n(\theta_{11}))(\ell_i > k)$.

Let m be the minimum such i . Since $n(P_m) \leq k < k+1$, and by the ordering we took, $\exists t_m, 1 \leq t_m \leq n(P_m) \leq k$ such that $s_{m,t_m} [f^{-1}(\theta_{11})] s_{m,k+1}$. Similarly, for each i such that $\ell_i > k$, $\exists t_i, 1 \leq t_i \leq n(P_i) \leq k$ such that $s_{i,t_i} [f^{-1}(\theta_{11})] s_{i,k+1}$.

Let $Q_1, \dots, Q_{k+1}$ be disjoint subsets of S defined as follows.

- (1) Let m be designated as above. Let $s_{m,k+1} \in Q_{k+1}$ and let $s_{m,1}, \dots, s_{m,k}$ be assigned to $Q_1, \dots, Q_k$ respectively.
- (2) For each $i, \ell_i > k$ and $i \neq m$, let $s_{i,k+1} \in Q_{k+1}$ and let $s_{i,t_i} \in Q_{t_i}$.
- (3) For each $t, t \neq t_i, 1 \leq t \leq k, \exists$ a unique $j \neq t_m, 1 \leq j \leq k \ni s_{i,t} \in Q_j$.
- (4) No two states $s_{i,t}, s_{i',t'}$ with $1 \leq t, t' \leq k$ belong to the same Q_j for any $j, 1 \leq j \leq k$.

Finally, each of the remaining states constitutes a set by

itself. Then there are exactly $r = \sum_{i=1}^{n(\theta_{11})} g(i)$ more sets,

denoted by $R_1, \dots, R_r$. Together with Q_i , they form a disjoint partition on S . This, in turn, induces an equivalence relation θ_{22} on S . Certainly $\theta_{11} \cap \theta_{22} = 0$ and $n(\theta_{22}) = (k+1) + \sum_{i=1}^{n(\theta_{11})} g(i)$.

Furthermore, define θ_{21} from θ_{11}, θ_{22} as in Definition 1.5. Thus we have

- (1) Every state in Q_{k+1} is θ_{21} -related to every state in Q_{t_m} by definition of Q_{k+1} and Q_{t_m} .
- (2) For each $j, 1 \leq j \leq r, \exists i, 1 \leq i \leq k \ni$ the only state $s \in R_j$ is θ_{21} related to states in Q_i .

(3) If $s \in Q_i$, $s' \in Q_{i'}$, and $i \neq i'$, $1 \leq i, i' \leq k$, then $s \notin \theta_{21} s'$.

Hence we have $n(\theta_{21}) = k$.

Q.E.D.

The following example shows that the number

$$n(\theta_{22}) = \begin{cases} k & \text{if } (\forall i)(\ell_i = k) \\ (k+1) + \sum_{i=1}^{n(\theta_{11})} g(i) & \text{otherwise} \end{cases}$$

may be the least upper bound.

Let $M = \langle S, T, f \rangle$ be given, where $S = \{1, 2, \dots, 8\}$, $T = \{\alpha\}$ and the transition table is as follows:

f	1	2	3	4	5	6	7	8
α	1	6	4	5	3	7	7	5

Let $\theta_{11} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$ and θ_{11} is of degree 2. In order to find $\theta_{22} \ni \theta_{11} \cap \theta_{22} = 0$, $n(\theta_{22})$ must be at least 4. However a choice of θ_{22} where $n(\theta_{22}) = 4$ will result in a $\theta_{21} \ni n(\theta_{21}) > 2$. In order to obtain $n(\theta_{21}) = 2$, θ_{22} may be $\{\{15\}, \{26\}, \{3\}, \{47\}, \{8\}\}$ and $n(\theta_{22}) = 5 = (k+1) + \sum_{i=1}^{n(\theta_{11})} g(i)$.

The results of this section are very important.

We shall summarize the significance as follows. Let $M = \langle S, T, f \rangle$ be a given automaton. If $\{M_1 : M_2\}$ is a GPD of M , then to find the number $n(\theta_{21})$, $n(\theta_{12})$ (input configurations from M_2 to M_1 and from M_1 to M_2 respectively) is a covering problem. Let θ_{11} be an equivalence relation on S . If k is the degree of θ_{11} , then k is the lower bound for $n(\theta_{21})$ for any choice of the equivalence relation θ_{22} such that $\theta_{11} \cap \theta_{22} = 0$. If k is the degree of θ_{11} and k is larger than the desired bound on the number of input configurations going into an automaton, then without finding the decomposition one may disregard θ_{11} .

immediately and pick another θ_{11}' such that θ_{11}' is of degree less than or equal to the bound. If θ_{11} is a non-trivial congruence relation (i.e., $1 < n(\theta_{11}) < n$) of degree k , then there always exists a non-trivial equivalence relation θ_{22} on S such that $\theta_{11} \wedge \theta_{22} = 0$, and $n(\theta_{21}) = k$. Finally, an algorithm can be easily obtained to find an equivalence relation θ_{22} on $S \ni$ (i) $\theta_{11} \wedge \theta_{22} = 0$ (ii) $n(\theta_{21}) = k$ and (iii) $n(\theta_{22}) = \text{minimum}$ of all $n(\theta_{22}')$, where θ_{22}' satisfies (i), (ii) (i.e., there exists a minimal state automaton M_2 such that $[M_1:M_2]$ is a GPD where M_1, M_2 induce θ_{11}, θ_{22} respectively and $n(\theta_{21}) = k$).

Chapter III

SET SYSTEMS AND *-COVERS

§1. *-covers with Substitution Property

Let $M = \langle S, T, f \rangle$ be a given incomplete automaton.

Definition 1.1. A set system $\mathcal{A} = (A_1, \dots, A_\ell)$ of S is a *-cover of S such that $(\forall i, j) (1 \leq i, j \leq \ell) [(A_i \subseteq A_j) \Leftrightarrow (i = j)]$.

Definition 1.2. A *-cover $\mathcal{A} = (A_1, \dots, A_\ell)$ of S is said to have the substitution property (SP) if and only if

$$(\forall t \in T) (\forall i) (1 \leq i \leq \ell) (\exists j) (1 \leq j \leq \ell)$$

$$[f(A_i, t) = \{f(s, t) \mid s \in A_i, (s, t) \in D(f)\} \subseteq A_j].$$

In the following, we shall show that the known results [12] for series or parallel decompositions are included in the results of Proposition 2.4 and Proposition 2.6 of the previous chapter.

Proposition 1.1. Let $M = \langle S, T, f \rangle$ be an incomplete automaton. Let $\mathcal{A}_1 = (A_1, \dots, A_\ell)$ be a *-cover with SP. Then there exists an automaton $M' = \langle \bar{S}, T, f' \rangle$, and a homomorphism ρ from M' to M and an ordinary congruence relation θ_{11} on M' such that θ_{11} induces $\mathcal{A}_1$ under ρ .

Proof. For each i , $1 \leq i \leq \ell$, let $A_i = \{s_{(i,1)}, \dots, s_{(i,p_i)}\}$. For each pair (i, j) such that $s_{(i,j)} \in A_i$, we associate a distinct new symbol $\bar{s}_{(i,j)}$. Let $\bar{A}_i = \{\bar{s}_{(i,1)}, \dots, \bar{s}_{(i,p_i)}\}$. Let $\bar{S}$ be the collection of all new symbols as i ranges over $\{1, 2, \dots, \ell\}$. Define a mapping ρ from $\bar{S}$ onto S such that for all $\bar{s}_{(i,j)} \in \bar{S}$, $\rho(\bar{s}_{(i,j)}) = s_{(i,j)}$. Since $\mathcal{A}_1$ has SP, for each $t \in T$, each coordinate A_i of $\mathcal{A}_1$, we choose a particular number $h(i, t) \in \{1, 2, \dots, \ell\}$ such that $f(A_i, t) \subseteq A_{h(i,t)}$. Let

f' be a function from $\bar{S} \times T$ into $\bar{S}$ such that if $f(s_{(i,j)}, t) = s_{(h(i,t), k)}$, then $f'(\bar{s}_{(i,j)}, t) = \bar{s}_{(h(i,t), k)}$. Let $M' = \langle \bar{S}, T, f' \rangle$. Since $f \circ \rho = \rho \circ f'$, ρ is a homomorphism from M' to M . The collection of subsets $\{\bar{A}_i\}$ forms a partition on $\bar{S}$, and thus induces a unique equivalence relation θ_{11} such that $[(\bar{s}_{(i,j)} \theta_{11} \bar{s}_{(h,k)}) \iff (i=k)]$. Certainly θ_{11} induces α_1 under ρ . It remains to show that θ_{11} is a congruence relation. Let $\bar{s}_{(i,j)} \theta_{11} \bar{s}_{(i,k)}$. For each t , by construction, $f'(\bar{s}_{(i,j)}, t) = \bar{s}_{(h(i,t), p)}$ and $f'(\bar{s}_{(i,k)}, t) = \bar{s}_{(h(i,t), q)}$. Hence $f'(\bar{s}_{(i,j)}, t) \theta_{11} f'(\bar{s}_{(i,k)}, t)$ and θ_{11} is a congruence relation.

Q.E.D.

As a result of this proposition and Proposition 2.4 of Chapter II, we have that if α_1 is a $*$ -cover of S with SP , then there exists a series decomposition for $M = \langle S, T, f \rangle$. Since a set system is a special $*$ -cover, our result in Proposition 2.4 includes the known results for set systems in [12].

Corollary 1.1. Let α be a $*$ -cover of S with SP . Let S'' be a set of new symbols. Suppose $\alpha'' = (A_1'', \dots, A_{\ell}'')$ is a partition on S'' such that there exists a function ρ'' from S'' onto S and such that α'' induces α under ρ'' . Suppose there exists a function γ from S'' onto $\bar{S}$ such that α'' induces α_1 under γ and $\gamma \circ \rho = \rho'$ where $\rho, \bar{S}, \alpha_1$, are as defined in Proposition 1.1. Then there exists a mapping from f'' on $S'' \times T$ into S'' such that $M'' = \langle S'', T, f'' \rangle$ realizes M and α'' is a congruence relation on M'' .

Proof. Simply define f'' as follows. For each $\bar{s}_{(i,j)} \in A_i''$, choose $f''(\bar{s}_{(i,j)}, t) \in \gamma^{-1}(f'(\gamma(s_{(i,j)}), t))$. It is easily shown that ρ' is exactly the homomorphism from M'' to M .

Q.E.D.

For the case of parallel decomposition, we have the following proposition. First, we introduce the following definition.

Definition 1.3. Let $\alpha = (A_1, \dots, A_\ell)$, $\beta = (B_1, \dots, B_m)$ be $*$ -covers of S . Then $(\alpha \cap \beta) = 0$ if and only if for each pair (i, j) , $1 \leq i \leq \ell$, $1 \leq j \leq m$, $(A_i \cap B_j)$ contains at most one element of S .

Proposition 1.2. Let $M = \langle S, T, f \rangle$. Let α_1, α_2 be two $*$ -covers of S with SP such that $(\alpha_1 \cap \alpha_2) = 0$. Then there exist an automaton $M' = \langle \bar{S}, T, f' \rangle$ and a homomorphism φ from M' to M and two ordinary congruence relations θ_{11}, θ_{22} on $\bar{S}$ such that $\theta_{11} \cap \theta_{22} = 0$, and θ_{11}, θ_{22} induce α_1, α_2 respectively under φ .

Proof. Let $\alpha_1 = (A_1, \dots, A_\ell)$, $\alpha_2 = (B_1, \dots, B_m)$. Let $E_{\alpha_1} = \{1, 2, \dots, \ell\}$, $E_{\alpha_2} = \{1, \dots, m\}$. Let $\mathcal{S} \subseteq E_{\alpha_1} \times E_{\alpha_2}$ such that $\mathcal{S} = \{(i, j) \mid (A_i \cap B_j) \neq \emptyset\}$. Define a mapping φ such that $\varphi(i, j) = s$ if $s \in (A_i \cap B_j) \neq \emptyset$. Thus $D(\varphi) = \mathcal{S}$. Since α_1, α_2 are $*$ -covers of S , $\pi_1(\mathcal{S}) = E_{\alpha_1}$, $\pi_2(\mathcal{S}) = E_{\alpha_2}$. Now consider φ as a table in which the rows are labeled by i , $1 \leq i \leq \ell$, and columns are labeled by j , $1 \leq j \leq m$. If $\varphi(i, j) = s$, then s occupies the entry (i, j) . For each occupied entry (i, j) , we introduce a new symbol $\bar{s}_{(i, j)}$. Let $\bar{S}$ be the collections of all new symbols. For each i , $1 \leq i \leq \ell$, let

$$\bar{A}_i = \{ \bar{s}_{(i, j)} \mid \text{the entry } (i, j) \text{ is occupied} \}.$$

For each j , $1 \leq j \leq m$, let $\bar{B}_j = \{ \bar{s}_{(i, j)} \mid \text{the entry } (i, j) \text{ is occupied} \}$. Let $\alpha_1' = (\bar{A}_1, \dots, \bar{A}_\ell)$, $\alpha_2' = (\bar{B}_1, \dots, \bar{B}_m)$. Thus $\{\bar{A}_i\}, \{\bar{B}_j\}$ are two partitions on $\bar{S}$ and hence induce two equivalence relations θ_{11}, θ_{22} on $\bar{S}$. Since $(\alpha_1 \cap \alpha_2) = 0$, by construction,

$\theta_{11} \cap \theta_{22} = 0$. It is obvious that θ_{11} , θ_{22} induce α_1 , α_2 respectively under φ .

By Corollary 1.1, for the $*$ -cover α_1 with SP, we have an automaton $M_1 = \langle \bar{S}, T, f_1 \rangle$ realizing M , and α_1' is a congruence relation on M_1 . Similarly, we have an automaton $M_2 = \langle \bar{S}, T, f_2 \rangle$ realizing M , and α_2' is a congruence relation on M_2 . Define an automaton $M' = \langle \bar{S}, T, f' \rangle$ as follows. For each $\bar{s}_{(i,j)} \in \bar{S}$ and each $t \in T$, define $f'(\bar{s}_{(i,j)}, t) = \bar{\xi}$, where $f_1(\bar{s}_{(i,j)}, t) \in A_{h_1}'(i, t)$, $f_2(\bar{s}_{(i,j)}, t) \in B_{h_2}'(j, t)$ and $\bar{\xi} = A_{h_1}'(i, t) \cap B_{h_2}'(j, t)$. We shall show that f' is well-defined on $\bar{S} \times T$ whenever f_1 and f_2 are defined, i.e., we must show $A_{h_1}'(i, t) \cap B_{h_2}'(j, t) \neq \emptyset$. By Corollary 1.1,

$$\varphi(f_1(\bar{s}_{(i,j)}, t)) \in A_{h_1}(i, t), \text{ and } \varphi(f_2(\bar{s}_{(i,j)}, t)) \in B_{h_2}(j, t).$$

Further,

$$\varphi(f_1(\bar{s}_{(i,j)}, t)) = f(s_{(i,j)}, t) = \varphi(f_2(\bar{s}_{(i,j)}, t)).$$

Hence, if $\xi = f(s_{(i,j)}, t)$, then ξ is the unique element in $A_{h_1}(i, t) \cap B_{h_2}(j, t)$ which implies $\bar{\xi}$ exists and is unique in $A_{h_1}'(i, t) \cap B_{h_2}'(j, t)$. Clearly α_1' is still a congruence relation on M' , since for each A_i' and $\bar{s}_{(i,j)} \in A_i'$, we have $f'(s_{(i,j)}, t) \in A_{h_1}'(i, t)$ where $f_1(\bar{s}_{(i,j)}, t) \in A_{h_1}'(i, t)$. Similarly, α_2' is a congruence relation on M' .

Q.E.D.

Again with the result of Proposition 2.6 of Chapter II, this includes the known results [12] for the parallel case.

Any attempt to obtain a stronger statement than that in Proposition 1.2 of this section, such as replacing the condition " $\alpha_1 \cap \alpha_2 = 0$ " by " (α, β) has ADDP" will not be successful. We show this by an example.

Example 1.1. Let $M = \langle S, T, f \rangle$, where $S = \{1, 2, \dots, 7\}$ and $T = \{\alpha\}$ and the transition table be as follows.

f	1	2	3	4	5	6	7
α	5	4	1	2	1	4	2

Let $\alpha_1 = (A_1, A_2, A_3)$ where $A_1 = \{4, 6\}$, $A_2 = \{2, 4, 7\}$, $A_3 = \{1, 3, 5\}$. Let $\alpha_2 = (B_1, B_2, B_3, B_4)$ where $B_1 = \{4, 5, 7\}$, $B_2 = \{1, 2\}$, $B_3 = \{3, 4\}$, and $B_4 = \{6\}$. It is easily shown that α_1, α_2 are set systems with SP, (α_1, α_2) has ADDP and $\alpha_1 \cap \alpha_2 \neq 0$. Let $\bar{\alpha}_1 = (\bar{A}_1, \bar{A}_2, \bar{A}_3)$ where $\bar{A}_1 = \{4, 6\}$, $\bar{A}_2 = \{2, 4', 7\}$, $\bar{A}_3 = \{1, 3, 5\}$ and let $\bar{\alpha}_2 = (\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4)$ where $\bar{B}_1 = \{4, 5, 7\}$, $\bar{B}_2 = \{1, 2\}$, $\bar{B}_3 = \{3, 4'\}$, $\bar{B}_4 = \{6\}$. Clearly $\bar{\alpha}_1, \bar{\alpha}_2$ are partitions on $\bar{S} = \{1, 2, 3, 4, 4', 5, 6, 7\}$ and $\bar{\alpha}_1 \cap \bar{\alpha}_2 = 0$. In order to make $\bar{\alpha}_1$ a partition with SP, we must require $f(4, \alpha) = 2$, $f(6, \alpha) = 4'$, $f(2, \alpha) = 4'$, $f(4', \alpha) = 2$, $f(7, \alpha) = 2$, $f(1, \alpha) = 5$, $f(3, \alpha) = 1$, $f(5, \alpha) = 1$. In this case, α_2 does not have SP any more, since $f(1, \alpha) = 4'$, $f(2, \alpha) = 5$, and $4', 5$ are not in one block of $\bar{\alpha}_2$.

§2. *-covers vs. Set Systems

Definition 2.1. Let $\alpha = (A_1, \dots, A_p)$ be a *-cover of S . Let $\alpha' = (A_1', \dots, A_p')$ be a set system of S . Then α' is said to

be induced by $\mathcal{A}$ is and only if

- (i) $(\forall i)(1 \leq i \leq p)(\exists j)(1 \leq j \leq p)[A_i \subseteq A_j]$
- (ii) $(\forall i)(1 \leq i \leq p)(\exists j)(1 \leq j \leq p)[A_i' = A_j]$

*-covers differ from set systems in that they may contain coordinates A_i, A_j where $A_i \subseteq A_j$ yet $i \neq j$. From the development, we see that each coordinate in a *-cover represents a state in an automaton. The natural question is then if $A_i \subseteq A_j$, do they represent the same state? Can we always replace the *-covers by their induced set systems and obtain essentially the same decomposition? In particular, if $[M_1:M_2]$ is a series or parallel decomposition of an automaton M , and $\mathcal{A}, \mathcal{B}$ are the *-covers that $[M_1:M_2]$ gives rise to, then if $\mathcal{A}, \mathcal{B}$ are replaced by their induced set systems, is the resultant still a series or parallel decomposition of M ?

We shall consider first the series decompositions. Let $M = \langle S, T, f \rangle$. Given $(\mathcal{A}, \mathcal{B})$ a pair of *-covers of S with ADDP such that $\mathcal{A}$ has SP. Then by Proposition 1.1 of Section 1, there exists a GPDT $\langle E_{\mathcal{A}}, E_{\mathcal{B}}, \varphi \rangle$ of M and a GPD $[M_1:M_2]$ such that it is a series decomposition of M , that is, there exists a GPD $[M_1:M_2]$ of M with $M' = G(M_1, M_2) = \langle S', T, f' \rangle$ such that $G(M_1, M_2)$ is a series decomposition and $\mathcal{A}$ induces a partition with SP on S' .

Theorem 2.1. Let $(\mathcal{A}, \mathcal{B})$ be a pair of *-covers of S such that $(\mathcal{A}, \mathcal{B})$ has ADDP. Let $\mathcal{A}$ have SP. Let $M' = \langle S', T, f' \rangle$ be defined as above. Then the induced set system $\mathcal{A}'$ has SP, and there exists a *-cover $\mathcal{B}'$ of S' such that

- (i) The number of coordinates of $\mathcal{B}'$ is no more than that of $\mathcal{B}$.
- (ii) $\langle E_{\mathcal{A}'}, E_{\mathcal{B}'}, \varphi' \rangle$ is a GPDT of M with an associated GPD

$[M_1':M_2']$ whose $G(M_1',M_2') = M''$ realizes M , and $[M_1':M_2']$ is a series decomposition of M .

Proof. Since α is a $*$ -cover with SP, by Proposition 1.1, there exists an automaton $M' = \langle S', T, f' \rangle$, a congruence relation $\bar{\alpha}$ on M' , an equivalence relation $\bar{\beta}$ on S' , and a homomorphism φ such that M' realizes M , and $\bar{\alpha}, \bar{\beta}$ induce α, β under φ respectively. Hence $\bar{\alpha}, \bar{\beta}$ give a partition pair series decomposition $[M_1':M_2']$ of M' . And $G(M_1':M_2')$ realizes M under φ .

Let α' be the set system induced by α . Clearly, α' is a set system with SP. Define β' from β such that $\beta' = (B_1', \dots, B_m')$ where for each j ,

$$B_j' = B_j - \{s \in S \mid (\exists i)(A_i \text{ in } \alpha)(A_i \text{ not in } \alpha')(\varphi(A_i, B_j) = s) \text{ and } (\nexists k)(A_k \text{ in } \alpha')(\varphi(A_k, B_j) = s)\}.$$

We note that B_j' may be empty. In such cases, instead of re-numbering the coordinates for β' , we will not have a j^{th} coordinate for β' . For each coordinate A_i' of α' and coordinate B_j' of β' (i.e., for each j , $B_j' \neq \emptyset$), define $\varphi'(A_i', B_j') = \varphi(A_i, B_j)$, if $(A_i, B_j) \in D(\varphi)$. Let S'' be the set of symbols obtained from the table of φ' . Then there exists an injection map β from S'' into S' such that $\varphi(\beta(\xi)) = \varphi'(\xi)$ for each $\xi \in S''$. Define $f''(\xi, t) \in \beta^{-1}(f'(\beta(\xi), t))$ where $(\xi, t) \in S'' \times T$ and $(\beta(\xi), t) \in D(f')$. Then $M'' = \langle S'', T, f'' \rangle$ clearly has a partition pair decomposition $[M_1':M_2']$ arising from α', β' , two partitions induced by α, β under φ' . Furthermore, φ' is a homomorphism from M'' to M .

Q. E. D.

Therefore, if (α, β) is a pair of $*$ -covers with ADDP such that α has SP, by replacing the $*$ -cover α by its induced

set system α' , there always exists a $*$ -cover β' such that α' , β' give rise to a series decomposition and β' has no more coordinates than β . However, the total number of connecting lines of the new series decomposition obtained from α', β' may be more than that of the original decomposition. We further note that not both α, β can be replaced by their induced set systems α', β' respectively, since α', β' may not have ADDP any more.

Let $M = \langle S, T, f \rangle$. Let α, β be two $*$ -covers with SP such that (α, β) give rise to a GPD $[M_1: M_2]$ of M where $G(M_1, M_2)$ is a parallel realization of M . We shall show by an example that even when α is already a set system, we still can not replace β by its induced set system β' and correspondingly α by a new set system α' such that the new resulting decomposition is still a parallel decomposition.

Example 2.1. Let $M = \langle S, T, f \rangle$ where $S = \{1, 2, \dots, 6\}$, $T = \{\alpha\}$ and the transition function f is given as follows:

f	1	2	3	4	5	6
α	5	4	1	2	1	2

Let $\alpha = (A_1, A_2, A_3)$ where $A_1 = \{4, 5, 6\}$, $A_2 = \{2, 1\}$, $A_3 = \{3, 4\}$.
 Let $\beta = (B_1, B_2, B_3)$ where $B_1 = \{1, 2, 4\}$, $B_2 = \{6, 2, 4\}$, $B_3 = \{1, 3, 5\}$.
 Let $E_\alpha = \{a_1, a_2, a_3\}$, $E_\beta = \{b_1, b_2, b_3\}$ denote α, β respectively.
 Let φ be a mapping from $E_\alpha \times E_\beta$ onto S as follows.

φ	b_1	b_2	b_3
a_1	4	6	5
a_2	2	2	1
a_3	4	4	3

Consider the specific GPD $\{M_1:M_2\}$ whose GPDT is $\langle E_\alpha, E_\beta, \varphi \rangle$ such that $G(M_1, M_2) = \langle \bar{S}, T, F \rangle$, where $\bar{S} = \{1, 2, 2', 3, 4, 4', 4'', 5, 6\}$ and

F	1	2	2'	3	4	4'	4''	5	6
α	5	4	4	1	2	2	2	1	2

Further, let $\bar{\alpha} = (\bar{A}_1, \bar{A}_2, \bar{A}_3)$, where $\bar{A}_1 = \{4, 5, 6\}$, $\bar{A}_2 = \{1, 2, 2'\}$, $\bar{A}_3 = \{3, 4', 4''\}$. Let $\bar{\beta} = (\bar{B}_1, \bar{B}_2, \bar{B}_3)$ where $\bar{B}_1 = \{2, 4, 4''\}$, $\bar{B}_2 = \{6, 2', 4'\}$, $\bar{B}_3 = \{1, 3, 5\}$. Then $\bar{\alpha}, \bar{\beta}$ are partitions with SP on $\bar{S}$ such that $\bar{\alpha} \cap \bar{\beta} = 0$. Hence this GPD $\{M_1:M_2\}$ is a parallel decomposition of M .

We note that β is not a set system, since $B_1 \subseteq B_2$. The induced set system then is $\beta' = (B_2', B_3')$, where $B_2' = \{6, 2', 4'\}$, $B_3' = \{1, 3, 5\}$. Then $\alpha' = (A_1', A_2', A_3')$ where $A_1' = \{5, 6\}$, $A_2' = \{1, 2'\}$, $A_3' = \{3, 4'\}$. Then α', β' are partitions on $S' = \{1, 2', 3, 4', 5, 6\}$. The transition function F' is

F'	1	2'	3	4'	5	6
α	5	4'	1	2'	1	2'

and φ' is defined by

φ'	b_2	b_3
a_1'	6	5
a_2'	2'	1
a_3'	4'	3

It can easily shown that f_{b_2}, f_{b_3} are two distinct non-compatible functions. Hence the resultant is not a paral-

lel decomposition of M any more.

In [12], Hartmanis and Stearns have studied series and parallel decompositions of an automaton by considering just set systems. We shall make some comment on the sufficiency of using just set systems. By the question of sufficiency we shall mean: are there series or parallel decompositions obtained by using $*$ -covers which are not obtainable from set systems? From our development, it seems to cast a strong doubt for the sufficiency. As illustrated before, a series decomposition using $*$ -cover with SP can also be obtained by its induced set system with SP. However, by so doing, the total number of input configurations may be increased. Since the number of input configurations is of great importance in the general theory, we can not be satisfied with decompositions using just set systems. Our Example 2.1 also demonstrates this point. Consider it as a series decomposition; by using the induced set system, the number of input configurations from M_2 to M_1 is increased from 1 to 2. However, this example does not answer the question of sufficiency completely. The desired example would be one for which there exists a decomposition using $*$ -covers having the minimum number of input configurations among all possible decompositions using set systems. At the present time, such an example has not been found. The trouble is mainly that the number of decompositions using set systems is very large and one has to resort to an exhaustive search. We thus leave this as an open question.

Chapter IV

MULTIPLE DECOMPOSITION

In this chapter, we extend our definition of pair decomposition to multiple decomposition. We shall show that the results for pair decomposition can be equally extended to the case of multiple decomposition. We shall further show that any multiple decomposition can be obtained by successive pairwise decompositions. Thus any discussion of multiple decomposition actually can be made through the discussion of pair decomposition.

Definition 1. Let $M = \langle S, T, f \rangle$ be a given deterministic, incomplete, finite automaton. Let $\{M_i = \langle S_i, T_i, f_i \rangle \mid 1 \leq i \leq N\}$ be a set of incomplete automata. Then $[M_1 : M_2 : \dots : M_N]$, denoted by $[M_i]_{1 \leq i \leq N}$, is a generalized N-tuple decomposition (GND) of M if and only if

(i) There exists a function φ defined on a subset

$$\mathcal{S} \subseteq \prod_{i=1}^N S_i \text{ onto } S \text{ such that for all } i, 1 \leq i \leq N,$$

$$\pi_i(\mathcal{S}) = S_i.$$

(ii) There exists a set of one to one functions

$\{h_i \mid 1 \leq i \leq N\}$ such that

$$(\forall i)(1 \leq i \leq N) [h_i : T_i \longrightarrow T \times \prod_{\substack{j=1 \\ j \neq i}}^N S_j].$$

$$(iii) (\forall i)(1 \leq i \leq N) D(f_i) = \{(s_i, t_i) \mid h_i(t_i) \\ = (t, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_N) \\ \ni [(\varphi(s_1, \dots, s_N), t) \in D(f)]\}$$

$$(iv) (\forall (s_1, \dots, s_N) \in \mathcal{S}) (\forall t \in T) [(\varphi(s_1, \dots, s_N), t) \in D(f)] \implies$$

$$(\forall i) (1 \leq i \leq N) \left[\left[\prod_{i=1}^N f_i(s_i, h_i^{-1}(t, s_1, \dots, s_{i-1}, \right.$$

$$\left. s_{i+1}, \dots, s_N)) \right] \in [\varphi^{-1}[f(\varphi(s_1, \dots, s_N, t))]] \right]$$

(v) Suppose $(s_1, \dots, s_N) \in \mathcal{S}$ and $\varphi(s_1, \dots, s_N) = f(s, t)$ for some $s \in S$, $t \in T$. Then

$$[\exists (s_1', \dots, s_N') \in \varphi^{-1}(s)] \left[(s_1, \dots, s_N) \right.$$

$$\left. = \left(\prod_{i=1}^N f_i(s_i', h_i^{-1}(t, s_1', \dots, s_{i-1}', s_{i+1}', \dots, s_N')) \right) \right]$$

Definition 2. Let GND tuple (GNDT) of an automaton $M = \langle S, T, f \rangle$ be an $(N+1)$ tuple $\langle S_1, S_2, \dots, S_N, \varphi \rangle$ where each S_i is a set of symbols, and φ is a mapping from $\mathcal{S} \subseteq \prod_{i=1}^N S_i$ onto S such that

$$(\forall i) [\pi_i(\mathcal{S}) = S_i].$$

The following results are similar to those in Chapter I. Since the proofs of these results are completely analogous to the corresponding ones in Chapter I, we shall omit them.

Theorem 1. Let $\{M_i = \langle S_i, T_i, f_i \rangle \mid 1 \leq i \leq N\}$ be a set of incomplete automata. If $\{M_i\}_{1 \leq i \leq N}$ is a GND of M , there exists a GNDT $\langle S_1, S_2, \dots, S_N, \varphi \rangle$ of M . Given a GNDT $\langle S_1, \dots, S_N, \varphi \rangle$ of M , there exists a non-empty collection of GNDs of M whose associated GNDT is the given $\langle S_1, \dots, S_N, \varphi \rangle$.

Theorem 2. For each GNDT $\langle S_1, S_2, \dots, S_N, \varphi \rangle$ of $M = \langle S, T, f \rangle$, and each ordering of the states in each $S_k = \{a_1^k, \dots, a_{\chi_k}^k\}$, there exists a set of $*$ -covers $\{\alpha_1, \dots, \alpha_N\}$ of S . Let A_i^k denote the i^{th} coordinate of the $*$ -covers α_k . For each fixed k , for each i ,

$$A_i^k = \{s \in S \mid (\exists (a_{i_1}^1, \dots, a_{i_{k-1}}^{k-1}, \dots, a_{i_N}^N)) (\varphi(s_{i_1}^1, \dots, s_{i_k}^k, \dots, s_{i_N}^N) = s)\}.$$

Definition 3. Let $\{\alpha_i \mid 1 \leq i \leq N\}$ be a set of $*$ -covers of S .

For each i , let $\alpha_i = (A_{i_1}, \dots, A_{i_{p_i}})$. Then $\{\alpha_i \mid 1 \leq i \leq N\}$

is said to be MD if and only if for each i , for each j^{th} entry, $A_{i,j} = \{s_1, \dots, s_{k_{i,j}}\}$ of α_i , there exists a

set of $k_{i,j}$ distinct $(N-1)$ -tuples such that

(i) For each k , $1 \leq k \leq k_{i,j}$, let the k^{th} $(N-1)$ -tuple be

$$(A_{1,j(i,k)}, \dots, A_{(i-1),j(i-1,k)}, A_{(i+1),j((i+1),k)}, \dots,$$

$$A_{N,j(N,k)}) \text{ where for each } q, 1 \leq q \leq N, q \neq i, \text{ we have } 1 \leq j(q,k) \leq p_q.$$

(ii) $s_q \in A_{q,j(q,k)}$ for each q and each k .

Definition 4. Let $\{\alpha_i \mid 1 \leq i \leq N\}$ be a set of MD $*$ -covers of S

such that $(\forall s)(\forall i, j) [m_{\alpha_i}(s) = m_{\alpha_j}(s) = m(s)]$. For each $s \in S$, $m(s) > 1$, $\{\alpha_i \mid 1 \leq i \leq N\}$ is said to have the first- s -deleted distributive property if and only if there exists a set $\{A_{1,j_1}, \dots, A_{N,j_N}\}$, where for each i , $1 \leq j_i \leq p_i$ and A_{i,j_i} is the j^{th} coordinate of α_i , such that

(i) $s \in A_{i,j_i}$ for each i .

(ii) $(\alpha_{1,((1,j_1),s)}, \dots, \alpha_{N,((N,j_N),s)})^1$ is again MD.

Definition 5. Let $\{\alpha_i \mid 1 \leq i \leq N\}$ be a set of MD $*$ -covers of S

such that $(\forall s)(\forall i, j) [m_{\alpha_i}(s) = m_{\alpha_j}(s) = m(s)]$. For each s , $m(s) > k$, $\{\alpha_i \mid 1 \leq i \leq N\}$ is said to have the k^{th} - s -deleted distributive property if and only if there exists

$$(A_{1,j_1}, \dots, A_{N,j_N}) \text{ such that}$$

¹Recall definition 2.4 of Chapter I for notation.

- (i) $s \in A_{i, j_i}$ for each i .
- (ii) $(\alpha_1, ((1, j_1), s), \dots, \alpha_N, ((N, j_N), s))$ has the $(k-1)^{\text{th}}$ - s -deleted distributive property.

Definition 6. Let $\{\alpha_i \mid 1 \leq i \leq N\}$ be a set of MD $*$ -covers of S such that $(\forall s)(\forall i)[m_{\alpha_i}(s) = m(s)]$. Then $\{\alpha_i \mid 1 \leq i \leq N\}$ is said to have the deleted distributive property (DDP) if for each s , $m(s) > 1$, $\{\alpha_i \mid 1 \leq i \leq N\}$ has the $(m(s)-1)^{\text{th}}$ - s -deleted distributive property.

Definition 7. Let $\{\alpha_i \mid 1 \leq i \leq N\}$ be a set of MD $*$ -covers of S . Then $\{\alpha_i \mid 1 \leq i \leq N\}$ is said to have the added deleted distributive property (ADDP) if and only if

- (i) There exists a set S' of symbols.
- (ii) There exists a set $\{\alpha_i' \mid 1 \leq i \leq N\}$ of $*$ -covers of S' .
- (iii) There exists a mapping ρ from S' onto S such that for each i , $1 \leq i \leq N$, α_i is induced by α_i' under ρ .

With all these definitions, we can show, by induction,

that there exists a mapping φ from a subset $\mathcal{S}$ of $\prod_{i=1}^N E_{\alpha_i}$ onto

S if and only if $\{\alpha_i \mid 1 \leq i \leq N\}$ has ADDP. The proof is similar to that in Chapter I. Hence, we arrive at a result similar to Theorem 2.4 of Chapter I.

Theorem 3. All GNDT of $M = \langle S, T, f \rangle$ are of the type

$\langle E_{\alpha_1}, \dots, E_{\alpha_N}, \varphi \rangle$ where E_{α_i} denotes the $*$ -cover α_i of S and $\{\alpha_i \mid 1 \leq i \leq N\}$ has the ADDP.

With an argument¹ similar to that in Section 1 of Chapter II, we shall simply study GND $\{M_i\}_{1 \leq i \leq N}$ of an automaton $M = \langle S, T, f \rangle$, where in its GNDT $\langle S_1, \dots, S_N, \varphi \rangle$, φ is an isomorphism from $\mathcal{S} \subseteq \prod_{i=1}^N S_i$ onto S , and $(\forall i)(\pi_i(\mathcal{S}) = S_i)$. For such a GND $\{M_i\}_{1 \leq i \leq N}$ of M , it naturally induces a set of N equivalence relations $\{\theta_{ii} \mid 1 \leq i \leq N\}$ on S as follows: For each i , $1 \leq i \leq N$, define θ_{ii} on S such that

$$(\forall s_1, s_2 \in S) [(s_1 \theta_{ii} s_2) \iff (\pi_i(\varphi^{-1}(s_1)) = \pi_i(\varphi^{-1}(s_2)))]$$

It can be easily shown that $\bigcap_{i=1}^N \theta_{ii} = 0$.

Definition 8. Let $\{\theta_{ii} \mid 1 \leq i \leq N\}$ be a collection of equivalence

relations on S such that $\bigcap_{i=1}^N \theta_{ii} = 0$. For each i , each j , $j \neq i$,

define a symmetric, reflexive relation θ_{ij} on S as follows:

For each $s_1, s_2 \in S$, $s_1 \theta_{ij} s_2$ if and only if

$$(\forall \alpha \in T)(\forall s_1')(\forall s_2') [(s_1' \theta_{ii} s_1)(s_2' \theta_{ii} s_2)(\forall k)(1 \leq k \leq N)(k \neq i)(s_1' \theta_{kk} s_2') \implies [f(s_1', \alpha) \theta_{jj} f(s_2', \alpha)]]$$

Theorem 4. Let $\{M_i = \langle S_i, T_i, f_i \rangle \mid i=1, 2, \dots, N\}$ be a set of incomplete automata. Let $\{M_i\}_{1 \leq i \leq N}$ be a GND of $M = \langle S, T, f \rangle$ such

that in the GNDT, φ is a one to one mapping from $\mathcal{S} \subseteq \prod_{i=1}^N S_i$

¹We recall and generalize that given a GND $\{M_i\}_{1 \leq i \leq N}$ of $M = \langle S, T, f \rangle$, we have an automaton $M' = \langle S', T, f' \rangle$ such that (1) M' realizes M , and (2) $\{M_i\}_{1 \leq i \leq N}$ is a GND of M' , (3) the homomorphism from $\mathcal{S} \subseteq \prod_{i=1}^N S_i$ onto S' is an isomorphism, (4) the properties of $\{M_i\}_{1 \leq i \leq N}$ as GND of M are completely represented by that of $\{M_i\}_{1 \leq i \leq N}$ as GND of M' .

onto S and for each i , $1 \leq i \leq N$, $\pi_i(\mathcal{S}) = S_i$. Then $\{M_i\}_{1 \leq i \leq N}$ gives rise to a set of relations $\{\theta_{ij} | 1 \leq i \leq N\}$ on S such that

(i) For each i , θ_{ii} is an equivalence relation on S

$$\text{and } \bigcap_{i=1}^N \theta_{ii} = 0.$$

(ii) $(\forall j)(\forall i) [\theta_{ij} \supseteq \theta_{ii}]$

(iii) $(\forall j) \left[\bigcap_{\substack{i=1 \\ i \neq j}}^N \theta_{ij} \subseteq f^{-1}(\theta_{jj}) \right].$

Proof. We need only show (ii) and (iii). Part (ii) can be easily shown. To prove (iii), let $s_1, s_2 \in S$ such that

$$s_1 \left[\bigcap_{\substack{i=1 \\ i \neq j}}^N \theta_{ij} \right] s_2. \quad \text{Then } (\forall s_1') (\forall s_2') \left[\left[s_1' \left(\bigcap_{i=1}^N \theta_{ii} \right) s_1' \right] \right. \\ \left. \left[s_2' \left(\bigcap_{i=1}^N \theta_{ii} \right) s_2' \right] \left[s_1' \left(\bigcap_{i=1}^N \theta_{ii} \right) s_2' \right] \implies \left[f(s_1', \alpha) \theta_{jj} f(s_2', \alpha) \right] \right].$$

Yet $\bigcap_{i=1}^N \theta_{ii} = 0$. Therefore, if $s_1 \left(\bigcap_{\substack{i=1 \\ i \neq j}}^N \theta_{ij} \right) s_2$ then $s_1 [f^{-1}(\theta_{jj})] s_2$.

Q.E.D.

Theorem 5. Let $M = \langle S, T, f \rangle$. Let $\{\theta_{ij}\}_{1 \leq i, j \leq N}$ be a set of relations on S such that

(i) For each i , θ_{ii} is an equivalence relation on S .

(ii) $\bigcap_{i=1}^N \theta_{ii} = 0$.

(iii) $(\forall j)(\forall i) [\theta_{ij} \supseteq \theta_{ii}]$

(iv) $(\forall j) \left[\bigcap_{\substack{i=1 \\ i \neq j}}^N \theta_{ij} \subseteq f^{-1}(\theta_{jj}) \right].$

Then there exists a set of incomplete automata $\{M_i = \langle S_i, T_i, f_i \rangle | i=1, 2, \dots, N\}$ such that

(i) $\{M_i\}_{1 \leq i \leq N}$ is a GND of M

(ii) In the GNDT $\langle S_1, \dots, S_N, \varphi \rangle$, φ is one to one from

$$\mathcal{S} \subseteq \prod_{i=1}^N S_i \text{ onto } S.$$

(iii) $\{M_i\}_{1 \leq i \leq N}$ gives rise to the set $\{\theta_{ij} \mid 1 \leq i, j \leq N\}$.

Proof. The proof is very similar to that of Theorem 1.2 of Chapter II. Q.E.D.

To show that any multiple decomposition can be obtained by successive pairwise decompositions, we consider the following approach to the formulation of GND of an automaton. Let

$M = \langle S, T, f \rangle$. Let $S_1, \dots, S_N$ be given sets such that there

exists a function from $\mathcal{S} \subseteq \prod_{i=1}^N S_i$ onto S such that for each

i , $\pi_i(\mathcal{S}) = S_i$. We may consider the transition function f

as a mapping from $T \times \prod_{i=1}^N S_i$ into $\prod_{i=1}^N S_i$, where f is a single

valued function on $D(f) = \mathcal{S} \subseteq T \times \prod_{i=1}^N S_i$, and f maps to $\emptyset$

otherwise. Let $\{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, N\}$. Define

$f_{i_1 i_2 \dots i_k} = f \circ P_{i_1 \dots i_k}$ such that $P_{i_1 \dots i_k}$ is the projection map from $\prod_{i=1}^N S_i$ on $\prod_{j=1}^k S_{i_j}$. Clearly, we may consider $f_{i_1 \dots i_k}$

as being defined on $T \times \prod_{j=k+1}^N S_{i_j}$ to $\prod_{j=1}^k S_{i_j}$ such that

$\{i_1, \dots, i_k, \dots, i_N\} = \{1, 2, \dots, N\}$. Let M_i be the automaton

with the set S_i of states, $T \times \prod_{\substack{j=1 \\ j \neq i}}^N S_j$ of inputs and f_i the

transition function. Then $f = (f_1, \dots, f_N)$. Thus, $[M_1 : \dots : M_N]$ is a GND of M and $G(M_1, \dots, M_N)$ is the automaton with the set $\prod_{i=1}^N S_i$ of states, set T of inputs and f the transition function.

In general, if $\{i_1, \dots, i_k\} \subseteq \{1, \dots, N\}$, then $G(M_{i_1}, \dots, M_{i_k})$ is the automaton with the set $\prod_{j=1}^k S_{i_j}$ of states, $T \times \prod_{\substack{h=1 \\ h \notin \{i_1, \dots, i_k\}}}^N S_h$ of inputs and $f_{i_1 \dots i_k}$ as the transition function. Since cartesian products are associative, we have

$$G(M_1, \dots, M_N) = G(G(M_{i_1}, \dots, M_{i_k}), M_{j_1}, \dots, M_{j_{N-k}})$$

where $\{i_1, \dots, i_k, j_1, \dots, j_{N-k}\} = \{1, \dots, N\}$. Hence any multiple decomposition can be obtained from successive pairwise decomposition. This result enables us to obtain results for N -tuple decomposition by successive use of the results for pair decomposition.

SUMMARY

This work is aimed at developing a theory of decomposition for deterministic, incomplete, finite automata allowing interconnections between component automata. In general, a decomposition for an automaton, even for a complete automaton, gives rise to a set of incomplete automata. Thus, in order to arrive at a theory allowing further decompositions, it is necessary to deal with incomplete automata instead of just complete automata.

Let $\mathcal{L}$ be a measure function for the complexity of an automaton such that if M' , M are two automata and M' realizes M , then $\mathcal{L}(M') \geq \mathcal{L}(M)$. Let B be a given bound for the complexity of each component automaton of a decomposition of $M = \langle S, T, f \rangle$. Then the final object of this work is to obtain an algorithm which generates all decompositions GND $\{M_1 : \dots : M_N\}$ of M such that

- (i) $(\forall i)(1 \leq i \leq N) [\mathcal{L}(M_i) \leq B]$
- (ii) $\sum_{\substack{i=1 \\ i \neq j}}^N \sum_{j=1}^N \lceil \log_2 n(\theta_{ij}) \rceil$ is a minimum.¹

¹We first refer to page 60 for the definition of θ_{ij} , $1 \leq i, j \leq N$. We remark that if one is interested, one may add an input term. In such a case, we may define two inputs $t_1, t_2 \in T$ as equivalent in an automaton M_i as follows: $(t_1 \omega_i t_2) \iff [(\forall s \in S)[f(s, t_1) \theta_{ii} f(s, t_2)]]$. Then clearly ω_i is a reflexive and symmetric relation on T . To find $n(\omega_i)$ is again a covering problem. If it is needed, one may add the term

We first considered decomposing a given automaton $M = \langle S, T, f \rangle$ into a pair of automata [Chapters I, II, III]. We then extended our theory to the multiple case and showed that any multiple decomposition can be obtained from successive pairwise decompositions [Chapter IV]. Therefore, in order to obtain an algorithm to find the desired GNDs, we need only study the properties of pair decomposition.

In Chapter I, we introduced the definitions of GPD of M and GPDT of M . We showed that any GPDT of M is of the type $\langle E_\alpha, E_\beta, \varphi \rangle$ where E_α, E_β are index sets of the $*$ -covers α, β of S respectively and (α, β) has ADDP. We studied the properties of GPD of M in Chapter II. We showed that in discussing the properties of a pair decomposition, it suffices to assume that all $*$ -covers are of the partition type. We deduced that to find the number of input configurations (thus the number of connecting lines) is a covering problem. We then introduced the definition of a congruence relation of degree k on the automaton M . We showed that each congruence relation of degree k on M induces a set of pair decompositions $[M_1 : M_1']$ such that M_1 has as its states the equivalence classes under θ . Furthermore, we proved that k is the lower bound on the number of input configurations from M_1' to M_1 in any induced decomposition $[M_1 : M_1']$ of M under θ . For any GPD $[M_1 : M_2]$ of M , it is considerably more involved for solving the covering problem to get the exact number of input configurations than

$\sum_{i=1}^N \lceil \log_2 n(\omega_i) \rceil$ to formula (ii), and require their sum to be a minimum.

determining the degree k of Θ which induces $[M_1:M_2]$. Thus in the desired algorithm we may first use the lower bound to check whether the complexity for a component automaton already exceeds the bound. Hence it eliminates much unnecessary searching for the desired GNDs. Suppose there exists a non-trivial congruence relation Θ_{11} of degree k_1 on M . Then we showed that there exists a non-trivial equivalence relation Θ_{22} on S such that if $[M_1:M_2]$ is the GPD induced by Θ_{11} , Θ_{22} , then the number of input configurations from M_2 to M_1 is exactly k_1 .

In Chapter III, we showed that our results include the known results for series and parallel decompositions of an automaton as special cases. We then raised the question of the sufficiency of set systems for considering decompositions for automata.

We conclude by listing two interesting yet open questions.

- (1) At present, we can only give an exhaustive algorithm for finding all the desired decompositions. Yet any exhaustive search procedure will not be satisfactory, since it is too slow. Thus, we have the following question.

Given an automaton M , does there exist a finite subcollection $F(M)$ of the collection $E(M)$ of all GPDs of M satisfying the following condition: For any $[M_1:M_2] \in E(M)$, there exists $[M_1':M_2'] \in F(M)$ such that M_1 realizes M_1' and M_2 realizes M_2' ?

This result will give a bound on the number of decompositions that an algorithm has to go through.

Partly to answer this question, the author has obtained an algorithm for generating all decompositions

without duplication [19] for those GPDs $[M_1:M_2]$ of M , where φ is an isomorphism in its GPDT $\langle S_1, S_2, \varphi \rangle$.

- (2) Find an example of an automaton M for which there exists a GND $[M_i]_{1 \leq i \leq N}$ obtained by using $*$ -covers such that

$$(i) (\forall i) [\mathcal{L}(M_i) \leq B]$$

- (ii) The number of interconnections is less than that of any decomposition of M which is obtained by using set systems on S , and which has the same upper bound B for the complexity of each component.

This result will indeed show that it is not sufficient to consider just set systems for decomposition of an automaton.

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